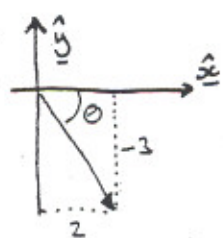
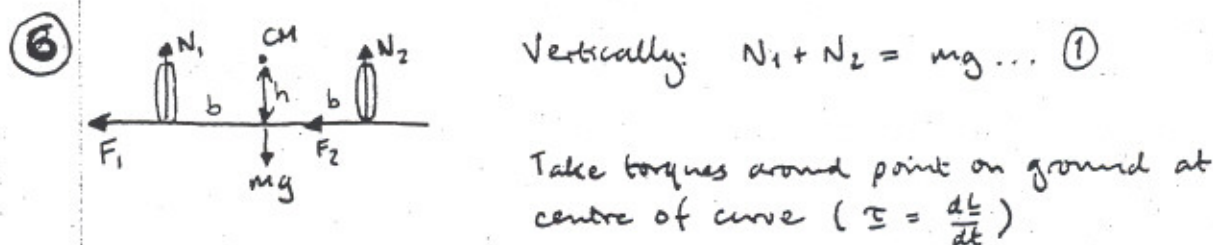


$$\underline{\tau} = aF \left[\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & 1 \\ 2 & -3 & 4 \end{vmatrix} \right]$$

$$= aF [\hat{x}(-1+3) - \hat{y}(1+4-2) + \hat{z}(2+1-3)] = aF (2\hat{x} - 3\hat{y})$$



$|\underline{\tau}| = \sqrt{13} aF$ making angle $\theta = \tan^{-1} \frac{3}{2} = 56.3^\circ$ with \hat{x} as shown.



Vertically: $N_1 + N_2 = mg \dots (1)$

Take torques around point on ground at centre of curve ($\underline{\tau} = \frac{dL}{dt}$)

$$N_1(r-b) + N_2(r+b) - mgr = h \frac{mv^2}{r} \dots (2)$$

↑ rate of change of ang. mom from centripetal accel.

Use (1) in (2) to give,

$$N_1 - N_2 = -\frac{h}{b} \frac{mv^2}{r} \dots (3)$$

Eliminate N_2 (assuming N_2 is normal force on outside wheel)

From (1) and (3)

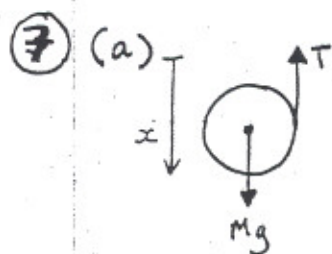
$$2N_1 = m \left(g - \frac{h}{b} \frac{v^2}{r} \right)$$

car will turn over if $N_1 < 0$, i.e. if $v^2 > \frac{gbr}{h}$, $v > \sqrt{\frac{gbr}{h}}$

For slipping: use radial eqn. of motion: $F_1 + F_2 = \frac{mv^2}{r}$

but: $F_1 + F_2 = \mu(N_1 + N_2) = \mu mg$, so slips when: $v^2 > \mu gr$

\therefore slips before turning over if $\mu gr < gbr/h \Rightarrow \mu < \frac{b}{h}$



linear: $Mg - T = M\ddot{x}$

angular (torques w.r.t. CM): $Ta = Mk^2\ddot{\omega}$

but: $a\ddot{\omega} = \ddot{x}$

Eliminate T: $\ddot{x} = \frac{g}{1 + k^2/a^2}$

and $T = \frac{Mgk^2}{k^2 + a^2}$

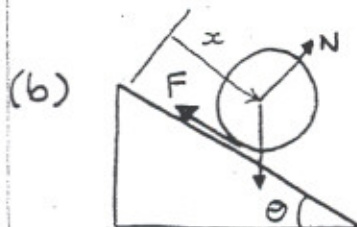
Alternatively use energy conservation.

$$\frac{1}{2}mv^2 + \frac{1}{2}Mk^2\omega^2 = Mgx$$

$$\frac{1}{2}v^2\left(1 + \frac{k^2}{a^2}\right) = gx$$

differentiate wrt time: $v\dot{v}\left(1 + \frac{k^2}{a^2}\right) = g\dot{x}v \Rightarrow \dot{v} = \frac{g}{1 + k^2/a^2}$

then use linear or angular eqns to get T.



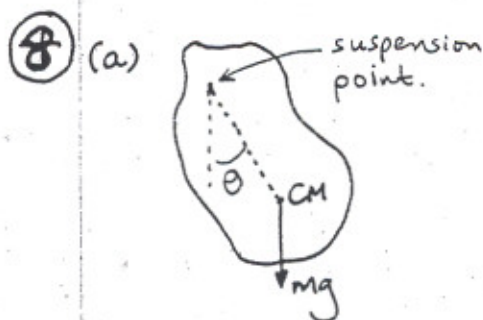
Friction force does no work: easiest to use energy conservation, so you don't have to eliminate F from eqns of motion.

$$\frac{1}{2}Mv^2 + \frac{1}{2} \cdot \frac{1}{2}Ma^2 \cdot \omega^2 - Mgx\sin\theta = 0$$

where x is distance rolled down slope. But $v = a\omega$ for rolling, so:

$$\frac{3}{4}v^2 = g\sin\theta x$$

differentiate wrt time: $\frac{3}{2}v\dot{v} = g\sin\theta v \Rightarrow \dot{v} = \frac{2}{3}g\sin\theta$

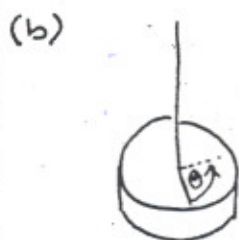


Use $\tau = \frac{dL}{dt}$ about suspension point

$$-mgl \sin\theta = I\ddot{\theta}$$

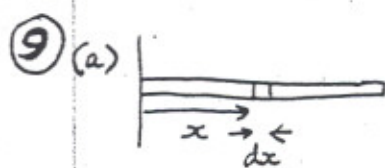
small $\theta \Rightarrow \ddot{\theta} = -\frac{mgl}{I}\theta$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$



Torque = $-\tau\theta$

so: $-\tau\theta = I\ddot{\theta} \Rightarrow T = 2\pi\sqrt{\frac{I}{\tau}}$



Total length a ; mass per unit length ρ

$$I = \int_0^a \rho dx \cdot x^2 = \frac{1}{3}\rho a^3 = \frac{1}{3}ma^2$$

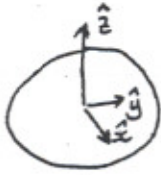


Radius a ; mass per unit area ρ .

Consider thin ring at radius r and integrate:

$$I = \int_0^a \rho \cdot 2\pi r dr \cdot r^2 = \frac{2\pi\rho a^4}{4} = \frac{1}{2}ma^2$$

(c)



Use perpendicular axis theorem:

$$I_x + I_y = I_z = \frac{1}{2}ma^2$$

But $I_x = I_y$ by symmetry $\therefore I_x = \frac{1}{4}ma^2$ is MoI about a diameter.(d) For thin spherical shell. Use xyz axes with origin at centre of shell.

$$I_x = \int dm (y^2 + z^2) \quad \text{and similarly for } I_y, I_z$$

\uparrow
mass element

But $I_x = I_y = I_z \equiv I$ by symmetry

$$\text{so: } 3I = I_x + I_y + I_z = \int dm 2(x^2 + y^2 + z^2) = 2ma^2$$

$$I = \frac{2}{3}ma^2$$

(e) Use similar argument for sphere:

$$3I = \int dm 2(x^2 + y^2 + z^2) \quad \text{but } dm = 4\pi r^2 dr \rho$$

where ρ is density.

$$3I = 2 \int_0^a 4\pi\rho r^2 \cdot r^2 dr = \frac{8\pi\rho a^5}{5}$$

$$I = \frac{8}{15}\pi\rho a^5 = \frac{2}{5}ma^2$$

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Isolated system \Rightarrow total ang. mom. is always zero.

$$I_1\omega_1 + I_2\omega_2 = 0$$

$$\Rightarrow \theta_2 = -\frac{I_1}{I_2}\theta_1 \quad \text{if } \theta_{1,2} \text{ are angular displacements.}$$

$$\text{During operation (a): } \phi = \theta_2 - \theta_1 = -\left(\frac{I_1 + I_2}{I_2}\right)\theta_1$$

$$\text{Similarly during (c): } -\phi = \theta_2' - \theta_1' = -\left(\frac{I_1' + I_2'}{I_2'}\right)\theta_1'$$

Total rotation is:

$$\theta_1 + \theta_1' = \phi \left[\frac{-I_2}{I_1 + I_2} + \frac{I_2'}{I_1' + I_2'} \right] = \boxed{\phi \left[\frac{I_2'/I_1' - I_2/I_1}{(1 + I_2/I_1)(1 + I_2'/I_1')} \right]}$$

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13

$$\text{Kinetic energy: } T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

$$\text{But } L = mr^2\dot{\theta} \text{ (ang. mom.) so } T = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2}$$

Potential energy: $V = -\frac{GMm}{r}$

$$\therefore E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

At r_{\max} and r_{\min} , $\dot{r} = 0$, so $E = \frac{L^2}{2mr_{\max}^2} - \frac{GMm}{r_{\max}} = \frac{L^2}{2mr_{\min}^2} - \frac{GMm}{r_{\min}}$

Eliminating L^2 gives: $r_{\max} + r_{\min} = -\frac{GMm}{E}$

For max and min speed: use $E = \frac{1}{2} m v^2 - \frac{GMm}{r}$

$\left. \begin{matrix} v_{\max} \\ v_{\min} \end{matrix} \right\}$ occurs for $\left. \begin{matrix} r_{\min} \\ r_{\max} \end{matrix} \right\}$

$$v_{\max} = \left[\frac{2}{m} \left(E + \frac{GMm}{r_{\min}} \right) \right]^{1/2} = \sqrt{2GM} \left(\frac{1}{r_{\min}} - \frac{1}{r_{\max} + r_{\min}} \right)^{1/2}$$

$$= \sqrt{2GM} \frac{r_{\max}}{\sqrt{r_{\min}(r_{\max} + r_{\min})}}$$

Similarly: $v_{\min} = \sqrt{2GM} \frac{r_{\min}}{\sqrt{r_{\max}(r_{\max} + r_{\min})}}$

Kepler's law is $T^2 = \frac{4\pi^2}{GM} a^3$ where $a = \frac{1}{2}(r_{\max} + r_{\min})$

Using $v_{\max} v_{\min} = \frac{2GM}{r_{\max} + r_{\min}}$ gives:

$$T^2 = 4\pi^2 \frac{2}{v_{\max} v_{\min} (r_{\max} + r_{\min})} \cdot \frac{(r_{\max} + r_{\min})^3}{8}$$

$$T = \frac{\pi(r_{\max} + r_{\min})}{v_{\max} v_{\min}}$$

(13) (i) $\frac{d}{dt}(\underline{r} \cdot \underline{v}) = 2\underline{r} \cdot \underline{v} = \frac{d}{dt}(r^2) = 2r \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{\underline{r} \cdot \underline{v}}{r}$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt} \left(\frac{\underline{r}}{r} \right) = \frac{\underline{v}}{r} - \frac{\underline{r}}{r^2} \frac{dr}{dt} = \frac{\underline{v}}{r} - \frac{\underline{r} \cdot \underline{v}}{r^3} \underline{r}$$

(ii) eqn motion: $\underline{\dot{p}} = -\frac{k}{r^3} \underline{r}$, so $\underline{\dot{p}} \times \underline{L} = -\frac{k}{r^3} \underline{r} \times (\underline{r} \times m\underline{v})$

$$= -\frac{mk}{r^3} \underline{r} \times (\underline{r} \times \underline{v})$$

(iii) $\underline{A} = \underline{p} \times \underline{L} - mk\hat{r}$

$$\underline{\dot{A}} = \underline{\dot{p}} \times \underline{L} - mk \frac{d\hat{r}}{dt} = -\frac{mk}{r^3} \underline{r} \times (\underline{r} \times \underline{v}) - \frac{mk}{r} \underline{v} + \frac{mk \underline{r} \cdot \underline{v}}{r^3} \underline{r}$$

$$\stackrel{\text{hint}}{=} -\frac{mk}{r^3} \underline{r} \cdot \underline{v} \underline{r} + \frac{mk}{r^3} r^2 \underline{v} - \frac{mk}{r} \underline{v} + \frac{mk}{r^3} \underline{r} \underline{v} \underline{r} = 0$$

$$(iv) \underline{r} \cdot \underline{A} = \underline{r} \cdot (\underline{p} \times \underline{L}) - mkr \stackrel{\text{hint}}{=} (\underline{r} \times \underline{p}) \cdot \underline{L} - mkr = L^2 - mkr$$

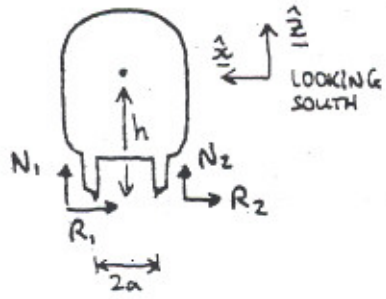
so: $rA \cos \theta = L^2 - mkr$

rearrange: $\frac{1}{r} = \frac{mk}{L^2} (1 + \frac{A}{mk} \cos \theta)$

comparing: $e = \frac{A}{mk}$ and $\ell = \frac{L^2}{mk}$

Direction of \underline{A} ? \underline{A} points towards the point of closest approach to the focus - e.g. \underline{A} points towards the perihelion for an orbit around the Sun.

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Coordinates: \hat{x} East, \hat{y} North, \hat{z} vertical

Eqn. motion: $\ddot{\underline{r}} = \frac{\underline{F}}{M} + \underline{g}^* - 2\underline{\omega} \times \dot{\underline{r}} - \underline{\omega} \times (\underline{\omega} \times \underline{r})$

Work to first order in ω ; so drop $\underline{\omega} \times (\underline{\omega} \times \underline{r})$ term, and use $\underline{g}^* = -g^* \hat{z} \approx -g \hat{z}$.

Also use $\dot{\underline{r}} = v \hat{y}$ and $\ddot{\underline{r}} = 0$

Earth's angular velocity, $\underline{\omega} = \omega \cos \lambda \hat{y} + \omega \sin \lambda \hat{z}$
 $\underline{\omega} \times \underline{v} = -\omega v \sin \lambda \hat{x}$

Resolve horizontally: $0 = -(R_1 + R_2)/M + 2\omega v \sin \lambda \dots (1)$

Resolve vertically: $0 = (N_1 + N_2)/M - g \dots (2)$

Torques about CM: $(R_1 + R_2)h + N_2 a - N_1 a = 0 \dots (3)$

Use (1) and (3): $N_1 - N_2 = \frac{2M\omega v h \sin \lambda}{a} \dots (4)$

Solve (2) and (4) for N_1 and N_2
 $N_1 = \frac{Mg}{2} (1 + \frac{2\omega v h \sin \lambda}{ga})$

$N_2 = \frac{Mg}{2} (1 - \frac{2\omega v h \sin \lambda}{ga})$

Magnitude of force on each rail is $(N_1^2 + R_1^2)^{1/2}$ or $(N_2^2 + R_2^2)^{1/2}$
But: (1) says $R_1 + R_2$ is order ω and we assume both of

R_1 and R_2 are $O(\omega)$. Therefore if $N_1 = \frac{Mg}{2}(1 + d\omega)$, we have,

$$(N_1^2 + R_1^2)^{1/2} = \frac{Mg}{2} (1 + 2\frac{d}{2}\omega) \text{ to order } \omega; \text{ likewise for } (N_2^2 + R_2^2)^{1/2}.$$

\therefore Ratio of forces is,
$$\frac{N_1}{N_2} = 1 + \frac{4\omega r h \sin \lambda}{ga} \text{ to order } \omega.$$

N_1 is larger, so larger force is on Eastward rail.

For $v = 150 \text{ km hr}^{-1}$ and $h = 2a$, we find:

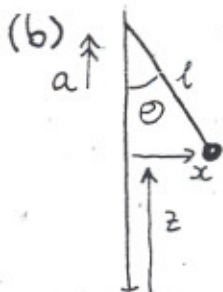
$$\frac{N_1}{N_2} = 1 + \frac{8\omega r h}{ga} \sin \lambda = 1 + \frac{8 \times \frac{2\pi}{24 \times 3600} \times \frac{150 \times 10^3}{3600} \text{ ms}^{-1}}{9.8 \text{ ms}^{-2}} \cdot \frac{1}{12}$$

$$= \boxed{1.00175}$$

16 (a) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ } $L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$
 $V = mgy$

(i) $\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0$ gives $\frac{d}{dt}(m\dot{x}) - 0 = 0$ or $\boxed{\ddot{x} = 0}$

(ii) $\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y} = 0$ gives $\frac{d}{dt}(m\dot{y}) - (-mg) = 0$ or $\boxed{\ddot{y} = -g}$



Let θ be generalised coordinate. Actual position of pendulum bob has:

$$x = l \sin \theta$$

$$z = z_0 + u_0 t + \frac{1}{2}at^2 - l \cos \theta$$

where z_0, u_0 are constants.

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}m(l^2\dot{\theta}^2 + (u_0 + at)^2 + 2(u_0 + at)l \sin \theta \dot{\theta})$$

$$V = mgz = mg(z_0 + u_0 t + \frac{1}{2}at^2 - l \cos \theta)$$

so, $L = T - V$ and:

$$\frac{\partial L}{\partial \theta} = m(u_0 + at)l \cos \theta \dot{\theta} - mgl \sin \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} + (u_0 + at)ml \sin \theta$$

Euler-Lagrange: $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 0$
 gives $ml^2 \ddot{\theta} + (u_0 + at)ml \cos \theta \dot{\theta} + aml \sin \theta - (u_0 + at)ml \cos \theta \dot{\theta} + mgl(\sin \theta) = 0$

$$ml^2 \ddot{\theta} + m(g+a)l \sin \theta = 0$$

For small oscillations: $\ddot{\theta} + \frac{(g+a)}{l} \theta = 0 \Rightarrow \boxed{T = 2\pi \sqrt{\frac{l}{g+a}}}$

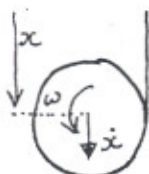
17 $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2} \cdot \frac{2}{5}ma^2\omega^2$; but $\dot{x} = a\omega \Rightarrow T = \frac{7}{10}m\dot{x}^2$

$$V = -max$$

$$L = T - V = \frac{7}{10}m\dot{x}^2 + max$$

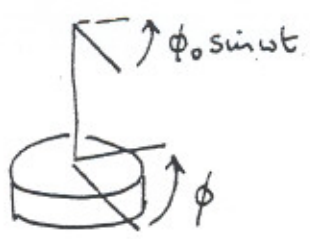
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0$ gives:

$$\frac{d}{dt}(\frac{7}{10}m\dot{x}) - mg = 0 \Rightarrow \boxed{\ddot{x} = \frac{5}{7}g}$$



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damping torque: $-k\dot{\phi}$
 restoring torque: $-T(\phi - \phi_0 \sin \omega t)$
 twist in fibre

angular eqn. of motion:
 $I\ddot{\phi} + k\dot{\phi} + T(\phi - \phi_0 \sin \omega t) = 0$

or: $\ddot{\phi} + \frac{k}{I}\dot{\phi} + \frac{T}{I}\phi = \frac{T}{I}\phi_0 \sin \omega t$

look for solution $Ae^{i\omega t}$ (A complex) with driving term $\frac{T}{I}\phi_0 e^{i\omega t}$ and take imaginary part at end [since $\text{Im}(\frac{T}{I}\phi_0 e^{i\omega t}) = \frac{T}{I}\phi_0 \sin \omega t$].

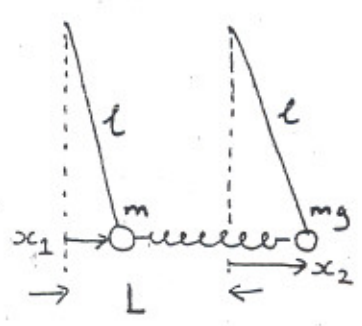
$$A[-\omega^2 + \frac{k}{I}i\omega + \frac{T}{I}] = \frac{T}{I}\phi_0 \Rightarrow A = \frac{T\phi_0}{I(-\omega^2 + \frac{k}{I}i\omega + \frac{T}{I})} = -i\sqrt{\frac{TI}{k^2}}\phi_0$$

\therefore Twist in fibre is $\phi - \phi_0 e^{i\omega t} = \phi_0(-i\sqrt{\frac{TI}{k^2}} - 1)e^{i\omega t}$

Imaginary part: $\phi_0[-\sqrt{\frac{TI}{k^2}} \cos \omega t - \sin \omega t]$

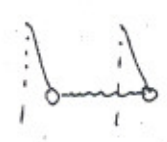
Max twist = amplitude = $\phi_0 \sqrt{1 + \frac{TI}{k^2}}$

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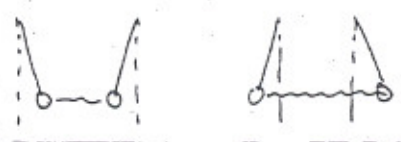


Expected form of normal modes:

(i) both swing together, spring neither stretched nor compressed.
 $(x_1 = x_2)$



(ii) swing in antiphase - spring alternately stretched + compressed
 $(x_1 = -x_2)$



Eqs. motion for small x_1, x_2

$$m\ddot{x}_1 = -mg\frac{x_1}{l} + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -mg\frac{x_2}{l} - k(x_2 - x_1)$$

$$\Rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -[\frac{g}{l} + \bar{k}] & \bar{k} \\ \bar{k} & -[\frac{g}{l} + \bar{k}] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

look for NORMAL MODES: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$

$$\bar{k} \equiv \frac{k}{m}$$

$$\Rightarrow -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -(\frac{g}{l} + \bar{k}) & \bar{k} \\ \bar{k} & -(\frac{g}{l} + \bar{k}) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Allowed values of ω^2 fixed by: $[\omega^2 - (\frac{g}{c} + \bar{k})]^2 - \bar{k}^2 = 0$

$$\omega^2 = \frac{2(\frac{g}{c} + \bar{k}) \pm \sqrt{4(\frac{g}{c} + \bar{k})^2 - 4(\frac{g}{c} + \bar{k})^2 + 4\bar{k}^2}}{2}$$

$$\omega^2 = \left(\frac{g}{c} + \bar{k} \right) \pm \bar{k}$$

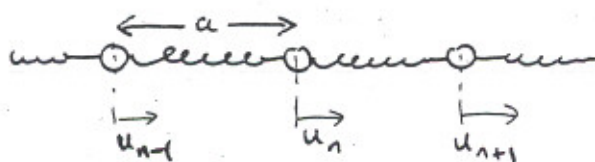
(i) $\omega^2 = \frac{g}{c}$ $\begin{pmatrix} -\bar{k} & \bar{k} \\ \bar{k} & -\bar{k} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \Rightarrow A_1 = A_2$

This corresponds to expected mode (i) above.

(ii) $\omega^2 = \frac{g}{c} + 2\bar{k}$ $\begin{pmatrix} \bar{k} & \bar{k} \\ \bar{k} & \bar{k} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \Rightarrow A_1 = -A_2$

expected mode (ii) above.

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$$T = \sum_n \frac{1}{2} m \dot{u}_n^2$$

$$V = \sum_n \frac{1}{2} \mu (u_n - u_{n-1})^2$$

$$L = \sum_n \frac{1}{2} m \dot{u}_n^2 - \sum_n \frac{1}{2} \mu (u_n - u_{n-1})^2$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_n} \right) = m \ddot{u}_n$$

$$\frac{\partial L}{\partial u_n} = -\mu(u_n - u_{n-1}) + \mu(u_{n+1} - u_n)$$

$$\Rightarrow \ddot{u}_n = \frac{\mu}{m} (u_{n-1} - 2u_n + u_{n+1})$$

Eqn. motion for infinite system. Look for translation invariant normal modes:

$$u_n = A e^{i(kna - \omega t)}$$

Substituting in eqn. motion gives dispersion relation:

$$\omega^2 = \frac{2\mu}{m} (1 - \cos(ka)) = \frac{4\mu}{m} \sin^2\left(\frac{ka}{2}\right)$$

For semi-infinite system: have u_0, u_1, u_2, \dots with $u_0 = h e^{-i\omega t}$ for some h . Try a combination:

$$u_n = \alpha e^{i(kna - \omega t)} + \beta e^{i(-kna - \omega t)}$$

Boundary condition on $u_0 \Rightarrow h = \alpha + \beta$. Physical requirement that only have wave propagating out along the line says $\beta = 0$.

$$\therefore u_n = h e^{i(kna - \omega t)} \text{ with } \omega = \omega_0 \sin\left(\frac{ka}{2}\right)$$

Case when $\omega > \omega_0$: dispersion relation requires $\cos(ka) < -1$, so ka must be complex. Let $\lambda = e^{ika}$ and solve for λ

$$\frac{1}{\lambda} + \lambda = 2 - \frac{4\omega^2}{\omega_0^2} < -2 \quad \text{Two solutions: } -1 < \lambda_1 < 0 \text{ and } \lambda_2 < -1$$

$|\lambda_2| > 1$ so this solution grows exponentially away from $n=0 \Rightarrow$ unacceptable
 \therefore use λ_1 with $-1 < \lambda < 0$ and write $u_n = h \lambda_1^n e^{-i\omega t}$
 [Oscillation exponentially damped in n . No propagating wave]

