# 3 The Liquid Drop Model

## 3.1 Some Nuclear Nomenclature

- **Nucleon:** A proton or neutron.
- **Atomic Number, Z:** The number of protons in a nucleus.
- **Atomic Mass number, A:** The number of nucleons in a nucleus.
- **Nuclide:** A nucleus with a specified value of A and Z. This is usually written as $^A_Z\{Ch\}$ where $Ch$ is the Chemical symbol. e.g. $^{56}_{28}$Ni means Nickel with 28 protons and a further 28 neutrons.
- **Isotope:** Nucleus with a given atomic number but different atomic mass number, i.e. different number of neutrons. Isotopes have very similar atomic and chemical behaviour but may have very different nuclear properties.
- **Isotone:** Nulceus with a given number of neutrons but a different number of protons (fixed (A-Z)).
- **Isobar:** Nucleus with a given A but a different Z.
- **Mirror Nuclei:** Two nuclei with odd A in which the number of protons in one nucleus is equal to the number of neutrons in the other and vice versa.

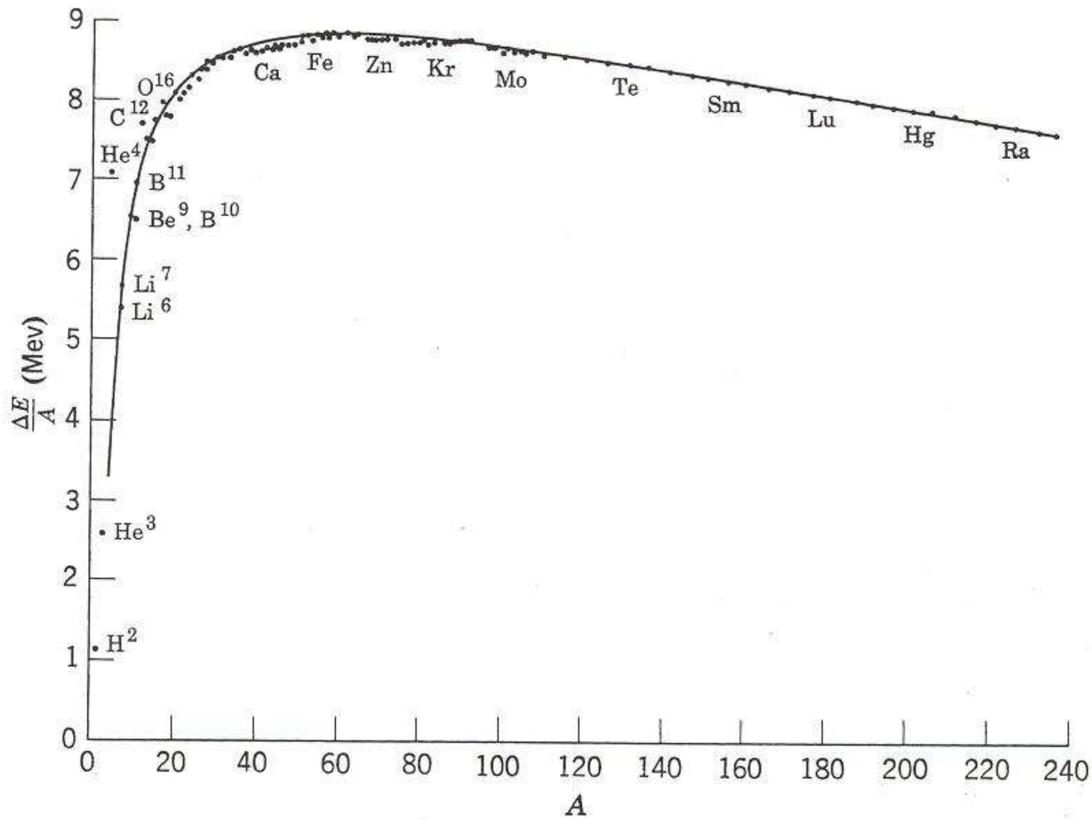
## 3.2 Binding Energy

The mass of a nuclide is given by

$$m_N \;=\; Z\,m_p + (A-Z)\,m_n \,-\, B(A,Z)/c^2,$$

where $B(A,Z)$ is the binding energy of the nucleons and depends on both Z and A. The binding energy is due to the strong short-range nuclear forces that bind the nucleons together. Unlike Coulomb binding these cannot even in principle be calculated analytically as the strong forces are much less well understood than electromagnetism.

Binding energies per nucleon increase sharply as A increases, peaking at iron (Fe) and then decreasing slowly for the more massive nuclei.




The binding energy divided by $c^2$ is sometimes known as the "mass defect".

## 3.3 Semi-Empirical Mass Formula

For most nuclei (nuclides) with $A > 20$ the binding energy is well reproduced by a semi-empirical formula based on the idea the the nucleus can be thought of as a liquid drop.

1. **Volume term:** Each nucleon has a binding energy which binds it to the nucleus. Therefore we get a term proportional to the volume i.e. proportional to A.

$$a_V\, A$$

   This term reflects the short-range nature of the strong forces. If a nucleon interacted with *all* other nucleons we would expect an energy term of proportional to $A(A-1)$, but the fact that it turns out to be proportional to A indicates that a nucleon only interact with its nearest neighbours.

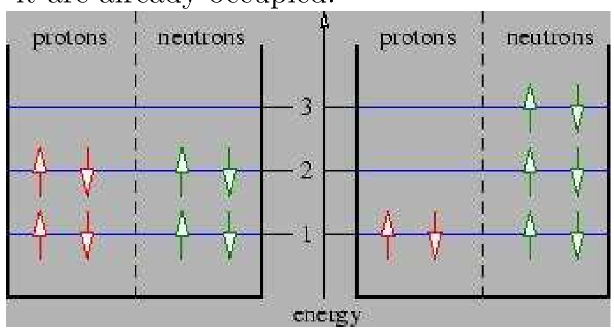
2. **Surface term:** The nucleons at the surface of the 'liquid drop' only interact with other nucleons inside the nucleus, so that their binding energy is reduced. This leads to a reduction of the binding energy proportional to the surface area of the drop, i.e. proportional to $A^{2/3}$

$$-a_S\, A^{2/3}.$$

3. **Coulomb term:** Although the binding energy is mainly due to the strong nuclear force, the binding energy is reduced owing to the Coulomb repulsion between the protons. We expect this to be proportional to the square of the nuclear charge, Z, ( the electromagnetic force is long-range so each proton interact with all the others), and by Coulomb's law it is expected to be inversely proportional to the nuclear radius, (the Coulomb energy of a charged sphere of radius $R$ and charge $Q$ is $3Q^2/(20\pi\epsilon_0 R)$) The Coulomb term is therefore proportional to $1/A^{1/3}$

$$-a_C \frac{Z^2}{A^{1/3}}$$

4. **Asymmetry term:** This is a quantum effect arising from the Pauli exclusion principle which only allows two protons or two neutrons (with opposite spin direction) in each energy state. If a nucleus contains the same number of protons and neutrons then for each type the protons and neutrons fill to the same maximum energy level (the 'fermi level'). If, on the other hand, we exchange one of the neutrons by a proton then that proton would be required by the exclusion principle to occupy a higher energy state, since all the ones below it are already occupied.




The upshot of this is that nuclides with $Z = N = (A-Z)$ have a higher binding energy, whereas for nuclei with different numbers of protons and neutrons (for fixed $A$) the binding energy decreases as the square of the number difference. The spacing between energy levels is inversely proportional to the volume of the nucleus - this can be seen by treating the nucleus as a three-dimensional potential well- and therefore inversely proportional to A. Thus we get a term

$$-a_A \frac{(Z-N)^2}{A}$$

5. **Pairing term:** It is found experimentally that two protons or two neutrons bind more strongly than one proton and one neutron.
In order to account for this experimentally observed phenomenon we add a term to the binding energy if number of protons and number of neutrons are *both* even, we subtract the same term if these are *both* odd, and do nothing if one is odd and the other is even. Bohr and Mottelson showed that this term was inversely proportional to the square root of the atomic mass number.
We therefore have a term

$$\frac{\left((-1)^Z + (-1)^N\right)}{2} \frac{a_P}{A^{1/2}}.$$

The complete formula is, therefore

$$B(A,Z) \;=\; a_V\,A \;-\; a_S\,A^{2/3} \;-\; a_C\frac{Z^2}{A^{1/3}} \;-\; a_A\frac{(Z-N)^2}{A} \;+\; \frac{\left((-1)^Z+(-1)^N\right)}{2}\frac{a_P}{A^{1/2}}$$

From fitting to the measured nuclear binding energies, the values of the parameters $a_V$, $a_S$, $a_C$, $a_A$, $a_P$ are

$$\begin{aligned} a_V &= 15.56 \text{ MeV} \\ a_S &= 17.23 \text{ MeV} \\ a_C &= 0.697 \text{ MeV} \\ a_A &= 23.285 \text{ MeV} \\ a_P &= 12.0 \text{ MeV} \end{aligned}$$

For most nuclei with $A > 20$ this simple formula does a very good job of determining the binding energies - usually better than 0.5%.

For example we estimate the binding energy per nucleon of $^{80}_{35}$Br (Bromine), for which Z=35, A=80 ($N = 80 - 35 = 45$) and insert into the above formulae to get

$$\begin{aligned} \text{Volume term:} &\quad (15.56 \times 80) \;=\; 1244.8\,\text{MeV} \\ \text{Surface term:} &\quad (-17.23 \times (80)^{2/3}) \;=\; -319.9\,\text{MeV} \\ \text{Coulomb term:} &\quad \left(\frac{0.697 \times 35^2}{(80)^{1/3}}\right) \;=\; -198.4\,\text{MeV} \\ \text{Asymmetry term:} &\quad \left(\frac{23.285 \times (45-35)^2}{80}\right) \;=\; -29.1\,\text{MeV} \\ \text{Pairing term:} &\quad \left(\frac{-12.0}{(80)^{1/2}}\right) \;=\; -1.3\,\text{MeV} \end{aligned}$$

Note that we *subtract* the pairing term since both (A-Z) and Z are odd. This gives a total binding energy of 696.1 MeV. The measured value is 694.2 MeV.

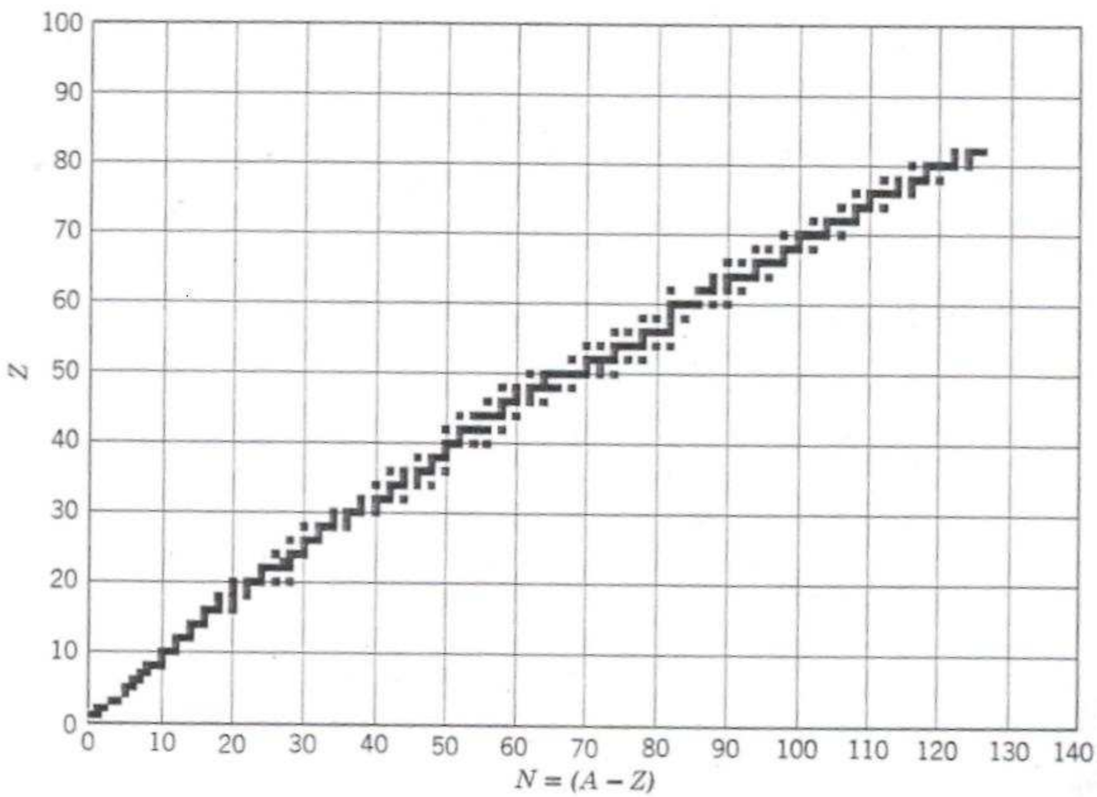
In order to calculate the mass of the nucleus we *subtract* this binding energy (divided by $c^2$) from the total mass of the protons and neutrons.

$$m_{Br} \;=\; 35m_p + 45m_n - 696.1 \;=\; 74423 \text{ Mev/c}^2.$$

Nuclear masses are nowadays usually quoted in MeV/c$^2$ but are still sometimes quoted in atomic mass units, defined to be 1/12 of the *atomic* mass of $^{12}_{6}$C (Carbon). The conversion factor is

$$1\,a.u. \;=\; 931.5 \text{ MeV/c}^2$$

Since different isotopes have different atomic mass numbers they will have different binding energies and some isotopes will be more stable than others. It turns out (and can be seen by looking for the most stable isotopes using the semi-empirical mass formula) that for the lighter nuclei the stable isotopes have approximately the same number of neutrons as protons, but above A $\sim$ 20 the number of neutrons required for stability increases up to about one and a half times the number of protons for the heaviest nuclei.




Qualitatively, the reason for this arises from the Coulomb term. Protons bind less tightly than neutrons because they have to overcome the Coulomb repulsion between them. It is therefore energetically favourable to have more neutrons than protons. Up to a certain limit this Coulomb effect beats the asymmetry effect which favours equal numbers of protons and neutrons.