

4 Nuclear Shell Model

4.1 Magic Numbers

The binding energies predicted by the Liquid Drop Model underestimate the actual binding energies of “magic nuclei” for which either the number of neutrons $N = (A - Z)$ or the number of protons, Z is equal to one of the following “magic numbers”

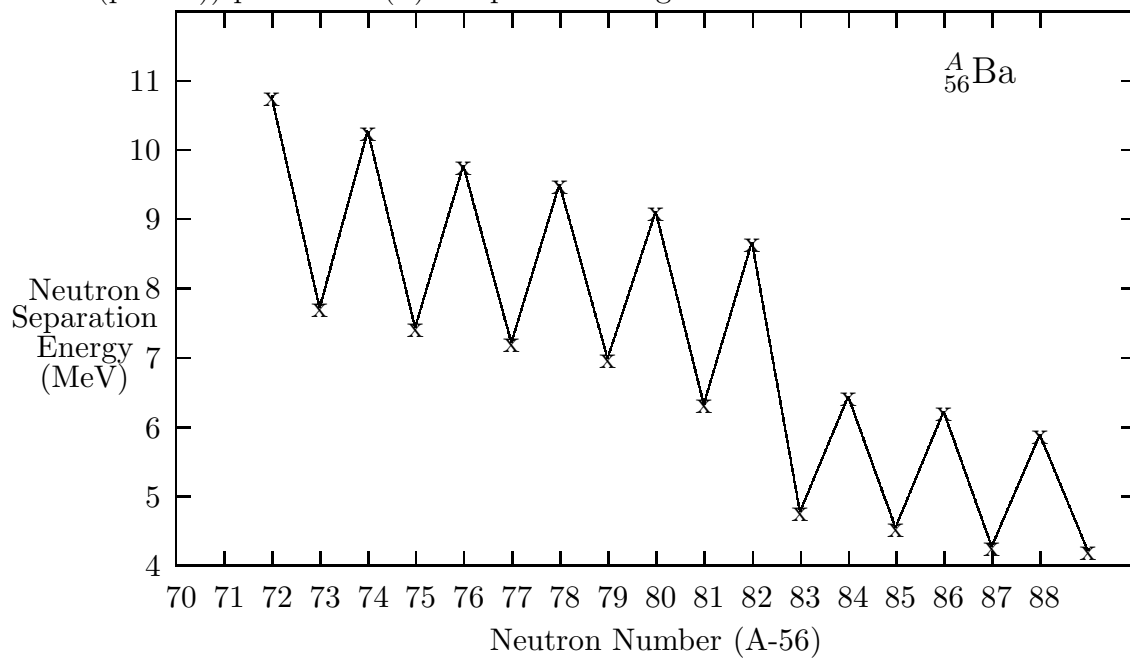
$$2, 8, 20, 28, 50, 82, 126.$$

This is particularly the case for “doubly magic” nuclei in which *both* the number of neutrons and the number of protons are equal to magic numbers.

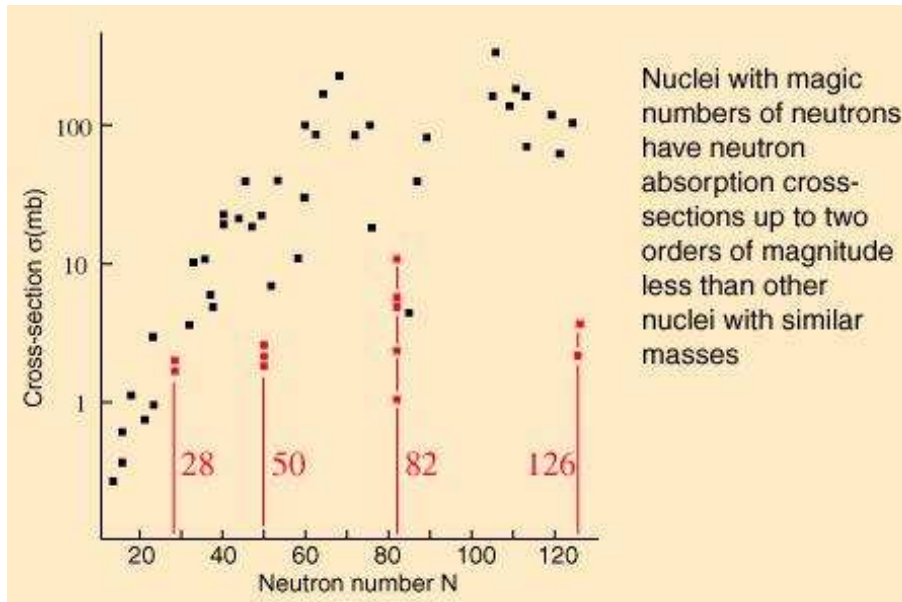
For example for $^{56}_{28}\text{Ni}$ (nickel) the Liquid Drop Model predicts a binding energy of 477.7 MeV, whereas the measured value is 484.0 MeV. Likewise for $^{132}_{50}\text{Sn}$ (tin) the Liquid Drop model predicts a binding energy of 1084 MeV, whereas the measured value is 1110 MeV.

There are other special features of magic nuclei:

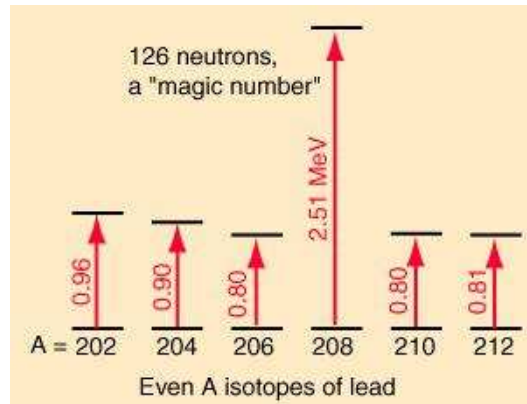
- The neutron (proton) separation energies (the energy required to remove the last neutron (proton)) peaks if N (Z) is equal to a magic number.



- There are more stable isotopes if Z is a magic number, and more stable isotones if N is a magic number.
- If N is magic number then the cross-section for neutron absorption is much lower than for other nuclides.



- The energies of the excited states are much higher than the ground state if either N or Z or both are magic numbers.



- Elements with Z equal to a magic number have a larger natural abundance than those of nearby elements.

4.2 Shell Model

These magic numbers can be explained in terms of the Shell Model of the nucleus, which considers each nucleon to be moving in some potential and classifies the energy levels in terms of quantum numbers $n l j$, in the same way as the wavefunctions of individual electrons are classified in Atomic Physics.

For a spherically symmetric potential the wavefunction (neglecting its spin for the moment) for any nucleon whose coordinates from the centre of the nucleus are given by polar coordinates (r, θ, ϕ) is of the form

$$\Psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \phi).$$

The energy eigenvalues will depend on the principle quantum number, n , and the orbital angular momentum, l , but are degenerate in the magnetic quantum number m . These energy levels come in ‘bunches’ called “shells” with a large energy gap just above each shell.

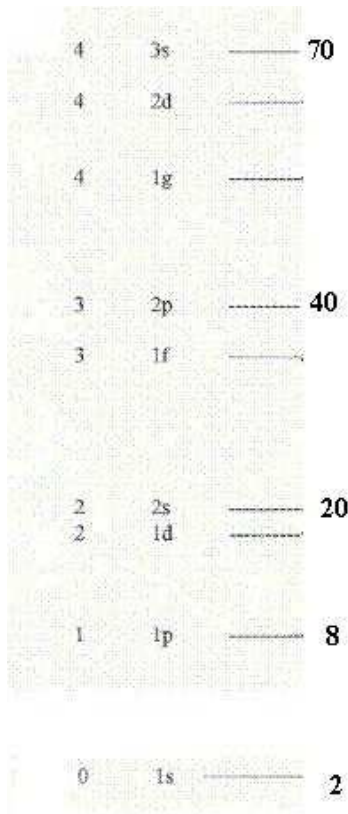
In their ground state the nucleons fill up the available energy levels from the bottom upwards with two protons (neutrons) in each available proton (neutron) energy level.

Unlike Atomic Physics we do not even understand in principle what the properties of this potential are - so we need to take a guess.

A simple harmonic potential (i.e. $V(r) \propto r^2$) would yield equally spaced energy levels and we would not see the shell structure and hence the magic numbers.

It turns out that once again the Saxon-Woods model is a reasonable guess, i.e.

$$V(r) = -\frac{V_0}{1 + \exp(((r - R)/\delta))}$$



For such a potential it turns out that the lowest level is $1s$ (i.e. $n = 1, l = 0$) which can contain up to 2 protons or neutrons. Then comes $1p$ which can contain up to a further 6 protons (neutrons). This explains the first 2 magic numbers (2 and 8). Then there is the level $1d$, but this is quite close in energy to $2s$ so that they form the same shell. This allows a further 2+10 protons (neutrons) giving us the next magic number of 20.

The next two levels are $1f$ and $2p$ which are also quite close together and allow a further 6+14 protons (neutrons). This would suggest that the next magic number was 40 - but experimentally it is known to be 50.

The solution to this puzzle lies in the spin-orbit coupling. Spin-orbit coupling - the interaction between the orbital angular momentum and spin angular momentum occurs in Atomic Physics. In Atomic Physics, the origin is magnetic and the effect is a small correction. In the case of nuclear binding the effect is about 20 times larger, and it comes from a term in the nuclear potential itself which is proportional to $\mathbf{L} \cdot \mathbf{S}$, i.e.

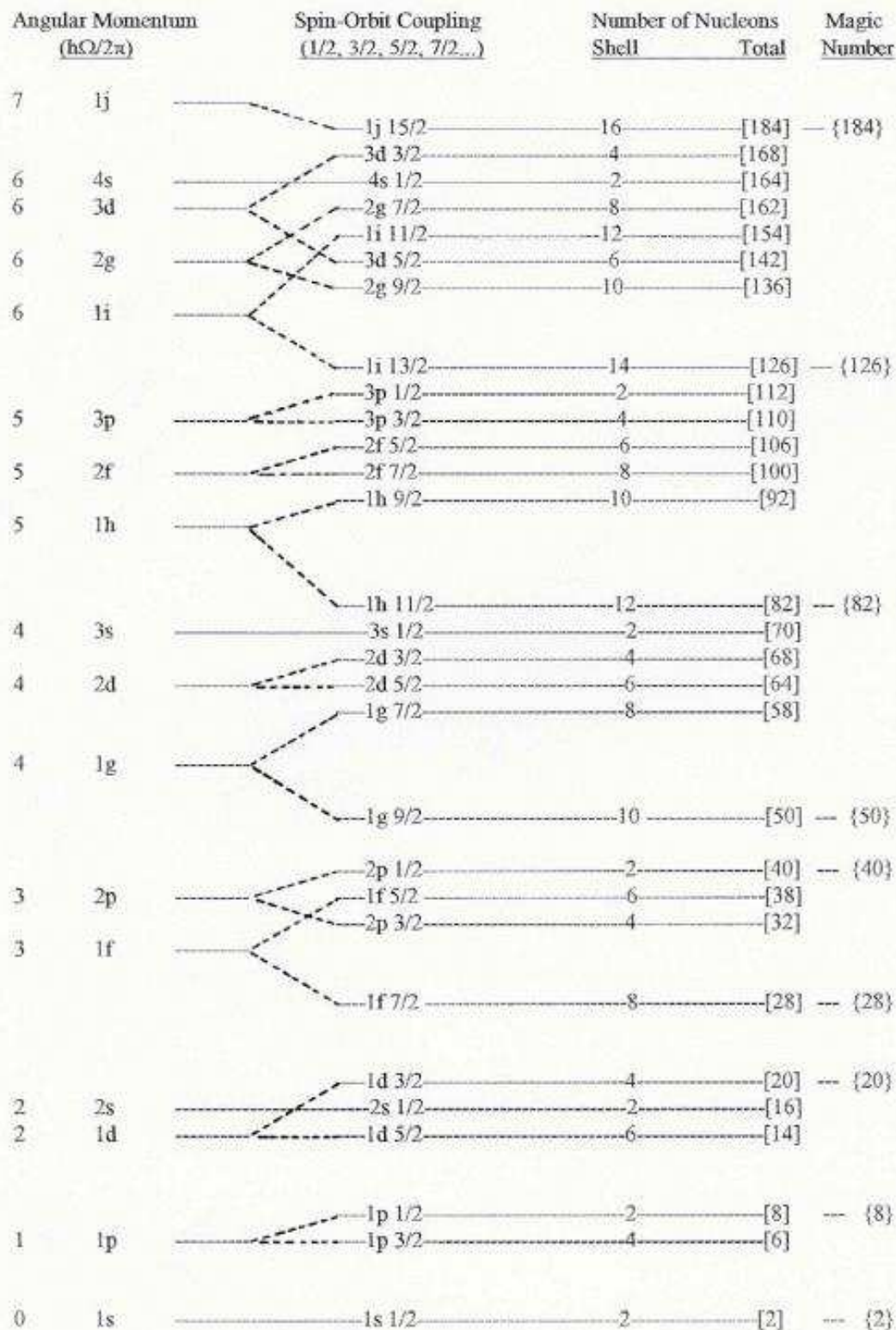
$$V(r) \rightarrow V(r) + W(r)\mathbf{L} \cdot \mathbf{S}.$$

As in the case of Atomic Physics (j-j coupling scheme) the orbital and spin angular momenta of the nucleons combine to give a total angular momentum j which can take the values $j = l + \frac{1}{2}$ or $j = l - \frac{1}{2}$. The spin-orbit coupling term leads to an energy shift proportional to

$$j(j+1) - l(l+1) - s(s+1), \quad (s = 1/2).$$

A further feature of this spin-orbit coupling in nuclei is that the energy split is in the opposite sense from its effect in Atomic Physics, namely that states with higher j have *lower energy*

Nuclear Shell Structure



We see that this large spin-orbit effect leads to crossing over of energy levels into different shells. For example the state above the $2p$ state is $1g$ ($l=4$), which splits into $1g_{\frac{9}{2}}$, ($j = \frac{9}{2}$) and $1g_{\frac{7}{2}}$, ($j = \frac{7}{2}$). The energy of the $1g_{\frac{9}{2}}$ state is sufficiently low that it joins the shell below, so that this fourth shell now consists of $1f_{\frac{7}{2}}$, $2p_{\frac{3}{2}}$, $1f_{\frac{5}{2}}$, $2p_{\frac{1}{2}}$ and $1g_{\frac{9}{2}}$. The maximum occupancy of this state ($(2j + 1)$ protons (neutrons) for each j) is now $8+4+6+2+10=30$, which added to the previous magic number, 20, gives the next observed magic number of 50.

Further up, it is the $1h$ state that undergoes a large splitting into $1h_{\frac{11}{2}}$ and $1h_{\frac{9}{2}}$, with the $1h_{\frac{11}{2}}$ state joining the lower shell.

4.3 Spin and Parity of Nuclear Ground States.

Nuclear states have an intrinsic spin and a well defined parity, $\eta = \pm 1$, defined by the behaviour of the wavefunction for all the nucleons under reversal of their coordinates with the centre of the nucleus at the origin.

$$\Psi(-\mathbf{r}_1, -\mathbf{r}_2 \cdots -\mathbf{r}_A) = \eta \Psi(\mathbf{r}_1, \mathbf{r}_2 \cdots \mathbf{r}_A)$$

The spin and parity of nuclear ground states can usually be determined from the shell model. Protons and neutrons tend to pair up so that the spin of each pair is zero and each pair has even parity ($\eta = 1$). Thus we have

- Even-even nuclides (both Z and A even) have zero intrinsic spin and even parity.
- Odd A nuclei have one unpaired nucleon. The spin of the nucleus is equal to the j -value of that unpaired nucleon and the parity is $(-1)^l$, where l is the orbital angular momentum of the unpaired nucleon.

For example ${}^{47}_{22}\text{Ti}$ (titanium) has an even number of protons and 25 neutrons. 20 of the neutrons fill the shells up to magic number 20 and there are 5 in the $1f_{\frac{7}{2}}$ state ($l = 3$, $j = \frac{7}{2}$) Four of these form pairs and the remaining one leads to a nuclear spin of $\frac{7}{2}$ and parity $(-1)^3 = -1$.

- Odd-odd nuclei. In this case there is an unpaired proton whose total angular momentum is j_1 and an unpaired neutron whose total angular momentum is j_2 . The total spin of the nucleus is the (vector) sum of these angular momenta and can take values between $|j_1 - j_2|$ and $|j_1 + j_2|$ (in unit steps). The parity is given by $(-1)^{(l_1+l_2)}$, where l_1 and l_2 are the orbital angular momenta of the unpaired proton and neutron respectively.

For example ${}^6_3\text{Li}$ (lithium) has 3 neutrons and 3 protons. The first two of each fill the $1s$ level and the third is in the $1p_{\frac{3}{2}}$ level. The orbital angular momentum of each is $l = 1$ so the parity is $(-1) \times (-1) = +1$ (even), but the spin can be anywhere between 0 and 3.

4.4 Magnetic Dipole Moments

Since nuclei with an odd number of protons and/or neutrons have intrinsic spin they also in general possess a magnetic dipole moment.

The unit of magnetic dipole moment for a nucleus is the “nuclear magneton” defined as

$$\mu_N = \frac{e\hbar}{2m_p},$$

which is analogous to the Bohr magneton but with the electron mass replaced by the proton mass. It is defined such that the magnetic moment due to a proton with orbital angular momentum \mathbf{l} is $\mu_N \mathbf{l}$.

Experimentally it is found that the magnetic moment of the proton (due to its spin) is

$$\mu_p = 2.79\mu_N = 5.58\mu_N s, \quad \left(s = \frac{1}{2}\right)$$

and that of the neutron is

$$\mu_n = -1.91\mu_N = -3.82\mu_N s, \quad \left(s = \frac{1}{2}\right)$$

If we apply a magnetic field in the z -direction to a nucleus then the unpaired proton with orbital angular momentum \mathbf{l} , spin \mathbf{s} and total angular momentum \mathbf{j} will give a contribute to the z - component of the magnetic moment

$$\mu^z = (5.58s^z + l^z) \mu_N.$$

As in the case of the Zeeman effect, the vector model may be used to express this as

$$\mu^z = \frac{(5.58 \langle \mathbf{s} \cdot \mathbf{j} \rangle + \langle \mathbf{l} \cdot \mathbf{j} \rangle)}{\langle \mathbf{j}^2 \rangle} j^z \mu_N$$

using

$$\begin{aligned} \langle \mathbf{j}^2 \rangle &= j(j+1)\hbar^2 \\ \langle \mathbf{s} \cdot \mathbf{j} \rangle &= \frac{1}{2} (\langle \mathbf{j}^2 \rangle + \langle \mathbf{s}^2 \rangle - \langle \mathbf{l}^2 \rangle) \\ &= \frac{\hbar^2}{2} (j(j+1) + s(s+1) - l(l+1)) \\ \langle \mathbf{l} \cdot \mathbf{j} \rangle &= \frac{1}{2} (\langle \mathbf{j}^2 \rangle + \langle \mathbf{l}^2 \rangle - \langle \mathbf{s}^2 \rangle) \\ &= \frac{\hbar^2}{2} (j(j+1) + l(l+1) - s(s+1)) \end{aligned} \tag{4.1}$$

We end up with expression for the contribution to the magnetic moment

$$\mu = \frac{5.58(j(j+1) + s(s+1) - l(l+1)) + (j(j+1) + l(l+1) - s(s+1))}{2j(j+1)} j \mu_N$$

and for a neutron with orbital angular momentum l' and total angular momentum j' we get (not contribution from the orbital angular momentum because the neutron is uncharged)

$$\mu = -\frac{3.82(j'(j'+1) + s(s+1) - l'(l'+1))}{2j'(j'+1)}j'\mu_N$$

Thus, for example if we consider the nuclide ${}^7_3\text{Li}$ for which there is an unpaired proton in the $2p_{\frac{3}{2}}$ state ($l = 1$, $j = \frac{3}{2}$) then the estimate of the magnetic moment is

$$\mu = \frac{5.58\left(\frac{3}{2} \times \frac{5}{2} + \frac{1}{2} \times \frac{3}{2} - 1 \times 2\right) + \left(\frac{3}{2} \times \frac{5}{2} + 1 \times 2 - \frac{1}{2} \times \frac{3}{2}\right)}{2 \times \frac{3}{2} \times \frac{5}{2}} \frac{3}{2} = 3.79\mu_N$$

The measured value is $3.26\mu_N$ so the estimate is not too good. For heavier nuclei the estimate from the shell model gets much worse.

The precise origin of the magnetic dipole moment is not understood, but in general they cannot be predicted from the shell model. For example for the nuclide ${}^{17}_9\text{F}$ (fluorine), the measured value of the magnetic moment is $4.72\mu_N$ whereas the value predicted from the above model is $-0.26\mu_N$. !! There are contributions to the magnetic moments from the nuclear potential that is not well-understood.

4.5 Excited States

As in the case of Atomic Physics, nuclei can be in excited states, which decay via the emission of a photon (γ -ray) back to their ground state (either directly or indirectly).

Some of these excited states are states in which one of the neutrons or protons in the outer shell is promoted to a higher energy level.

However, unlike Atomic Physics, it is also possible that sometimes it is energetically cheaper to promote a nucleon from an inner closed shell, rather than a nucleon from an outer shell into a high energy state. Moreover, excited states in which more than one nucleon is promoted above its ground state is much more common in Nuclear Physics than in Atomic Physics.

Thus the nuclear spectrum of states is very rich indeed, but very complicated and cannot be easily understood in terms of the shell model.

Most of the excited states decay so rapidly that their lifetimes cannot be measured. There are some excited states, however, which are metastable because they cannot decay without violating the selection rules. These excited states are known as “isomers”, and their lifetimes can be measured.

4.6 The Collective Model

The Shell Model has its shortcomings. This is particularly true for heavier nuclei. We have already seen that the Shell Model does not predict magnetic dipole moments or the spectra

of excited states very well.

One further failing of the Shell Model are the predictions of electric quadrupole moments, which in the Shell Model are predicted to be very small. However, heavier nuclei with A in the range 150 - 190 and for $A > 220$, these electric quadrupole moments are found to be rather large.

The failure of the Shell Model to correctly predict electric quadrupole moments arises from the assumption that the nucleons move in a spherically symmetric potential.

The Collective Model generalises the result of the Shell Model by considering the effect of a non-spherically symmetric potential, which leads to substantial deformations for large nuclei and consequently large electric quadrupole moments.

One of the most striking consequences of the Collective Model is the explanation of low-lying excited states of heavy nuclei. These are of two types

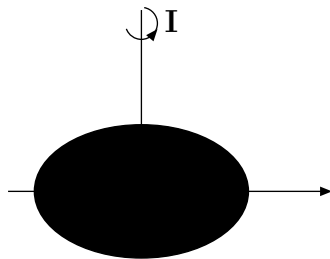
- **Rotational States:** A nucleus whose nucleon density distributions are spherically symmetric (zero quadrupole moment) cannot have rotational excitations (this is analogous to the application of the principle of equipartition of energy to monatomic atoms for which there are no degrees of freedom associated with rotation).

On the other hand a nucleus with a non-zero quadrupole moment can have excited levels due to rotational perpendicular to the axis of symmetry.

For an even-even nucleus whose ground state has zero spin, these states have energies

$$E_{\text{rot}} = \frac{I(I+1)\hbar^2}{2\mathcal{I}}, \quad (4.2)$$

where \mathcal{I} is the moment of inertia of the nucleus about an axis through the centre perpendicular to the axis of symmetry.



It turns out that the rotational energy levels of an even-even nucleus can only take even values of I . For example the nuclide ${}_{72}^{170}\text{Hf}$ (hafnium) has a series of rotational states with excitation energies

$$E \text{ (KeV)} : \quad 100, \quad 321, \quad 641$$

These are almost exactly in the ratio $2 \times 3 : 4 \times 5 : 6 \times 7$, meaning that these are states with rotational spin equal to 2, 4, 6 respectively. The relation is not exact because the moment of inertia changes as the spin increases.

We can extract the moment of inertia for each of these rotational states from eq.(4.2). We could express this in SI units, but more conveniently nuclear moments of inertia are quoted in $\text{MeV}/c^2 \text{fm}^2$, with the help of the relation

$$\hbar c = 197.3 \text{ MeV fm.}$$

Therefore the moment inertia of the $I = 2$ state, whose excitation energy is 0.1 MeV, is given (insetting $I = 2$ into eq.(4.2) by

$$\mathcal{I} = 2 \times 3 \times \frac{\hbar^2 c^2}{2c^2 E_{\text{rot}}} = \frac{6 \cdot 197.5^2}{2 \cdot 0.1} = 1.17 \times 10^6 \text{ MeV}/c^2 \text{fm}^2$$

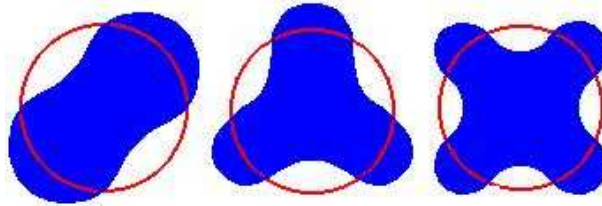
For odd-A nuclides for which the spin of the ground state I_0 is non-zero, the rotational levels have excitation levels of

$$E_{\text{rot}} = \frac{1}{2\mathcal{I}} (I(I+1) - I_0(I_0+1)) \hbar^2,$$

where I can take the values $I_0 + 1$, $I_0 + 2$ etc. For example the first two rotational excitation energies of $^{143}_{60}\text{Nd}$ (neodymium), whose ground state has spin $\frac{7}{2}$, have energies 128 KeV and 290 KeV. They correspond to rotational levels with nuclear spin $\frac{9}{2}$ and $\frac{11}{2}$ respectively. The ratio of these two excitation energies (2.27) is almost exactly equal to

$$\frac{\frac{11}{2} \times \frac{13}{2} - \frac{7}{2} \times \frac{9}{2}}{\frac{9}{2} \times \frac{11}{2} - \frac{7}{2} \times \frac{9}{2}} = 2.22$$

- **Shape oscillations:** These are modes of vibration in which the deformation of the nucleus oscillates - the electric quadrupole moment oscillates about its mean value. It could be that this mean value is very small, in which case the nucleus is oscillating between an oblate and a prolate spheroidal shape. It is also possible to have shape oscillations with different shapes



The small oscillations about the equilibrium shape perform simple harmonic motion. The energy levels of such modes are equally spaced. Thus an observed sequence of equally spaced energy levels within the spectrum of a nuclide is interpreted as a manifestation of such shape oscillations.