Lepton Mixing and Cancellation of the Dirac Mass Hierarchy in SO(10) GUTs with Flavor Symmetries

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based on
C. Hagedorn, MS, A. Yu. Smirnov PRD **79**, 036002 (2009) [arXiv:0811.2955 [hep-ph]]

Outline

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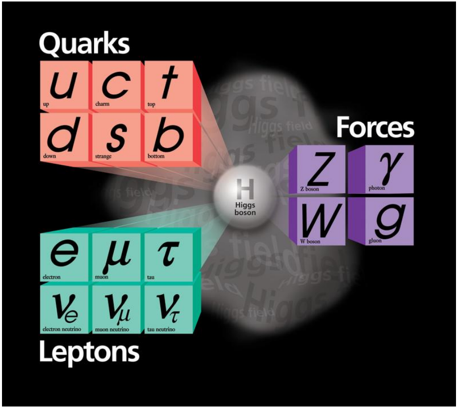
What do we know?



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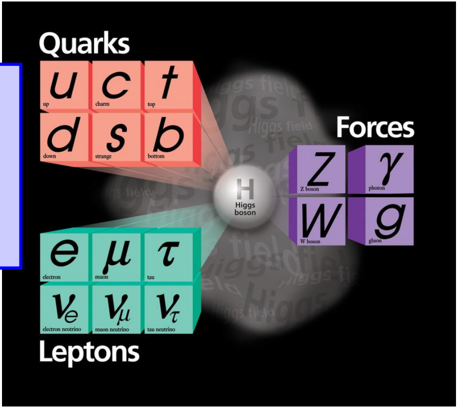

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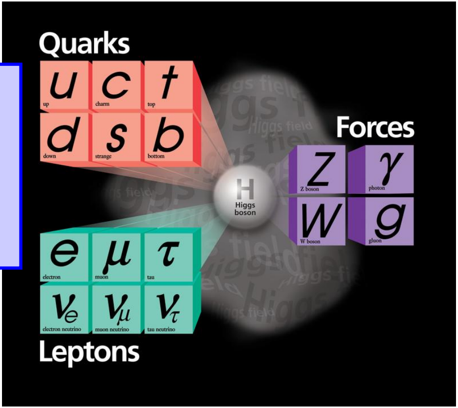

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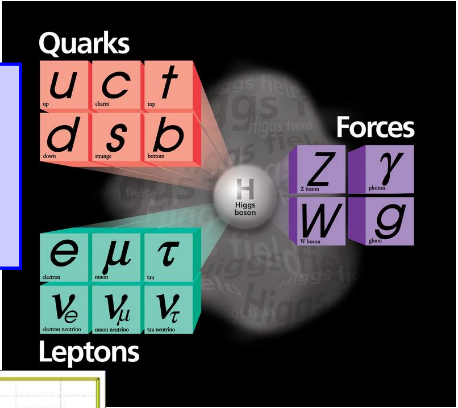

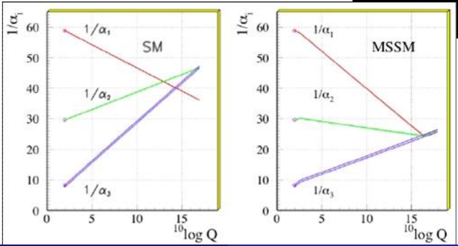

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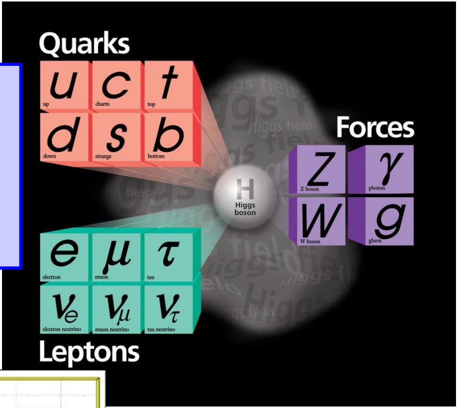
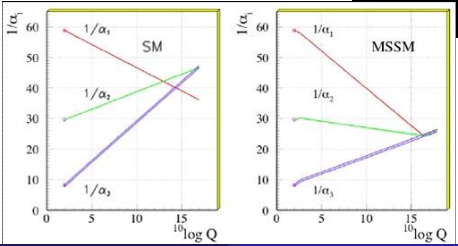
$\Rightarrow$ Grand unified theory?





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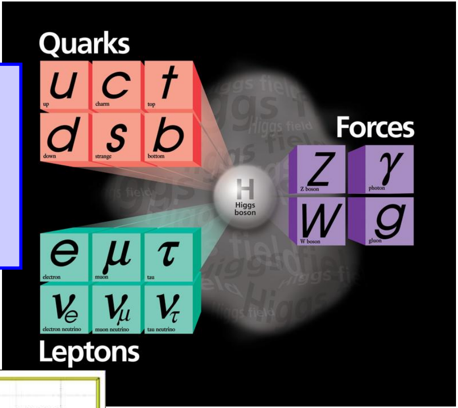
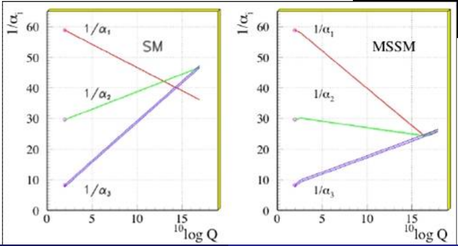
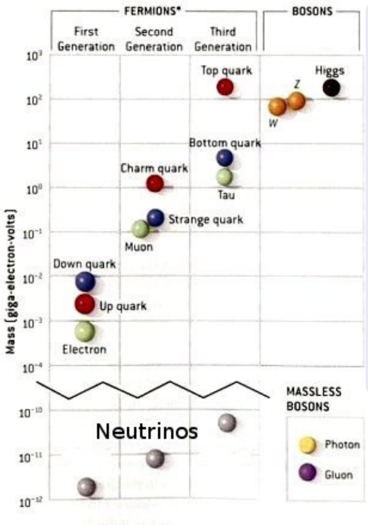




Why are there three families?

Fermion Masses



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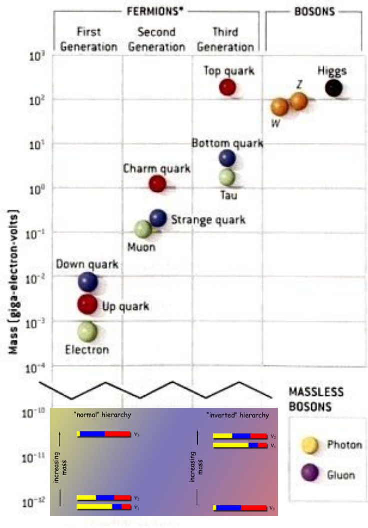
Fermion Masses



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- Neutral fermions have smaller masses and a weaker hierarchy:
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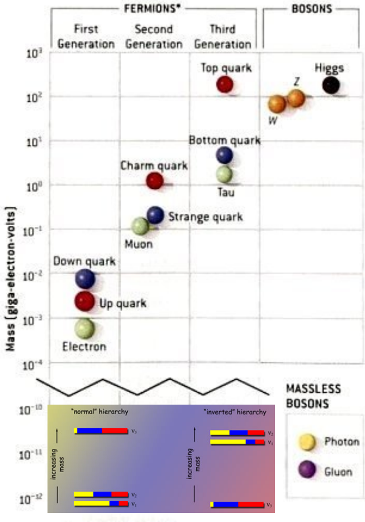
Fermion Masses



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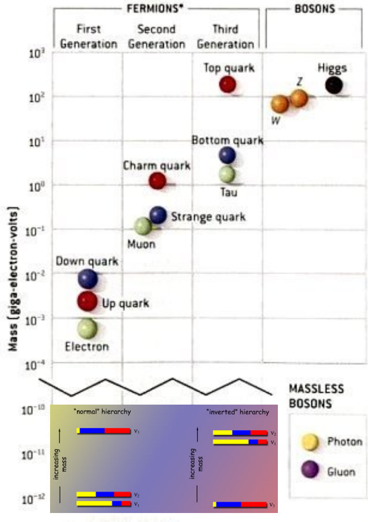

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  $\theta_{12} \approx 34.4^\circ$, $\theta_{23} \approx 45^\circ$, $\theta_{13} \lesssim 12.8^\circ$
  $\Rightarrow$ compatible with tri-bimaximal mixing:
  $\sin^2\theta_{12} = \frac{1}{3}$, $\sin^2\theta_{23} = \frac{1}{2}$, $\sin^2\theta_{13} = 0$

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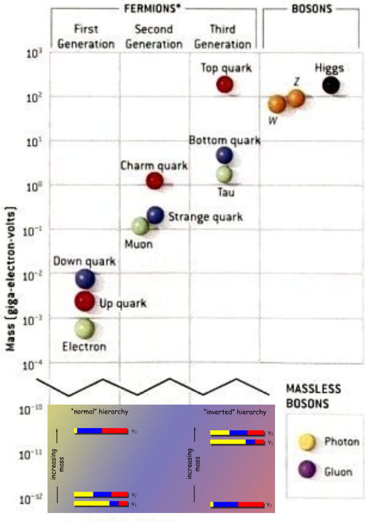

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- Explanation of different structures?

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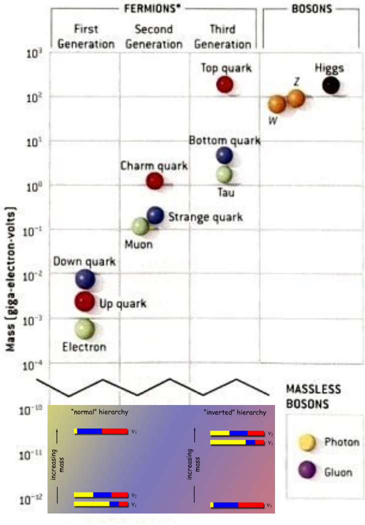

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- Explanation of different structures?
- Compatibility with GUTs?

Seesaw Mechanism

Standard Seesaw [Minkowski;Yanagida;Glashow;Gell-Mann,Ramond,Slansky;Mohapatra,Senjanovic]

Introduction of right-handed neutrinos N

$$\mathcal{M} = \begin{pmatrix} 0 & m_D \\ \cdot & M_{NN} \end{pmatrix} \Rightarrow m_\nu \approx -m_D M_{NN}^{-1} m_D^T$$

$(m_D \sim \mathcal{O}(\Lambda_{ew}) \ , \quad M_{NN} \sim \mathcal{O}(\Lambda_{B-L}))$
leads to effective mass of light neutrinos $m_\nu \lesssim \mathcal{O}(1\,\mathrm{eV})$

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- Different flavor structure possible
- But in SO(10): $m_D \sim m_u$
  $\Rightarrow$ Large (quadratic) hierarchy in neutrino masses
- Cancellation of hierarchies needed in neutrino mass matrix

Outline

Cancellation Mechanism

Cascade Seesaw [Mohapatra,Valle;Barr]

- Additional singlets S

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & m_{\nu S} \\ . & 0 & M_{NS} \\ . & . & M_{SS} \end{pmatrix}$$

$$m_D, m_{\nu S} \sim \mathcal{O}\left(\Lambda_{ew}\right) \;, \quad M_{NS} \sim \mathcal{O}\left(\Lambda_{GUT}\right) \;, \quad M_{SS} \sim \mathcal{O}\left(M_{Pl}\right)$$

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$$\mathcal{M} = \begin{pmatrix} 0 & m_D & m_{\nu S} \\ . & 0 & M_{NS} \\ . & . & M_{SS} \end{pmatrix} \Rightarrow m_\nu = m_\nu^{DS} + m_\nu^{LS}$$

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- Double seesaw (DS) contribution: $m_\nu^{DS} \approx m_D {M_{NS}^{-1}}^T M_{SS} M_{NS}^{-1} m_D^T$
- Linear seesaw (LS) contribution: $m_\nu^{LS} \approx -\left[m_D {M_{NS}^{-1}}^T m_{\nu S}^T + (\dots)^T\right]$
  $\Rightarrow$ generally smaller

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Cancellation [Smirnov]

- $F \equiv m_D {M_{NS}^{-1}}^T$ non hierarchical $\Rightarrow$ hierarchy of m_ν given by M_{SS}

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- $F \equiv m_D M_{NS}^{-1\,T}$ non hierarchical $\Rightarrow$ hierarchy of m_ν given by M_{SS}
- $F \propto 1$ (Dirac screening $\Rightarrow$ Dirac flavor structure is cancelled)

Setup within SO(10)

Lagrangian

$$\alpha_{ij}\, \underline{\mathbf{16}}_i\, \underline{\mathbf{16}}_j\, H + \beta_{ij}\, \underline{\mathbf{16}}_i \Delta\, S_j + (M_{SS})_{ij}\, S_i S_j$$

	$\underline{\mathbf{16}}_i$	S_i	H	Δ
SO(10)	$\underline{\mathbf{16}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{10}}$	$\underline{\overline{\mathbf{16}}}$

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & m_{\nu S} \\ . & 0 & M_{NS} \\ . & . & M_{SS} \end{pmatrix}$$

$$m_D = \alpha \langle H \rangle \;, \quad M_{NS} = \beta \langle \Delta \rangle_N \;, \quad m_{\nu S} = \beta \langle \Delta \rangle_\nu$$

$\Rightarrow$ Correlation between α and β needed.

Which Flavor Symmetry?

Lagrangian

- Explain number of generations: $\underline{\mathbf{16}}_i \sim \underline{\mathbf{3}}$

Particle Content

	$\underline{\mathbf{16}}_i$	S_i	H	Δ	χ_i
SO(10)	$\underline{\mathbf{16}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{10}}$	$\overline{\underline{\mathbf{16}}}$	$\underline{\mathbf{1}}$

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$\Sigma(81)$	$\underline{\mathbf{3}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{i}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{3}}_\mathbf{2} \cong \underline{\mathbf{3}}_\mathbf{1}^\star$

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- Explain difference in CKM and MNS matrix (in lowest order)
- Flavons (gauge group singlets charged with respect to G_F) χ

$$\frac{\alpha_{ij}}{\Lambda}\underline{\mathbf{16}}_i\,\underline{\mathbf{16}}_j\,H\chi^{(\star)} + \frac{\beta_{ij}}{\Lambda}\underline{\mathbf{16}}_i\,\Delta\,S_j\chi + (M_{SS})_{ij}\,S_iS_j\ ,$$

Particle Content

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T_7: Group Theory

- $T_7 \cong Z_7 \rtimes Z_3 \subset \mathrm{SU}(3)$, also called Frobenius group
- Smallest group with complex $\underline{\mathbf{3}}$: order 21
- Irreducible representations: $\underline{\mathbf{1}}_\mathbf{1}$, $\underline{\mathbf{1}}_\mathbf{2}$, $\underline{\mathbf{1}}_\mathbf{3} \cong \underline{\mathbf{1}}_\mathbf{2}^*$ and $\underline{\mathbf{3}}$, $\underline{\mathbf{3}}^\star$
- $\underline{\mathbf{1}}_\mathbf{i}$ like in Z_3: $\underline{\mathbf{1}}_\mathbf{1}$ and $\underline{\mathbf{1}}_\mathbf{2} \otimes \underline{\mathbf{1}}_\mathbf{3}$ are invariant
- Generators of $\underline{\mathbf{3}}$: $(A^7 = \mathbf{1},\, B^3 = \mathbf{1},\, A^3\, B\, A^2 = B)$

$$A = \begin{pmatrix} e^{2\pi\mathrm{i}/7} & 0 & 0 \\ 0 & e^{4\pi\mathrm{i}/7} & 0 \\ 0 & 0 & e^{8\pi\mathrm{i}/7} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- $\underline{\mathbf{3}} \otimes \underline{\mathbf{1}}_\mathbf{i} = \underline{\mathbf{3}}$:
  $(a_1,\, a_2,\, a_3)^T \sim \underline{\mathbf{3}}$, $c \sim \underline{\mathbf{1}}_\mathbf{i}$:
  $$(a_1\, c, \omega^{i-1}\, a_2\, c, \omega^{1-i}\, a_3\, c) \text{ with } \omega = e^{\mathrm{i}2\pi/3}$$
- $\{\underline{\mathbf{3}} \otimes \underline{\mathbf{3}}\} = \underline{\mathbf{3}} \oplus \underline{\mathbf{3}}^*$:
  $(a_1,\, a_2,\, a_3)^T \sim \underline{\mathbf{3}}$: $\quad (a_3\, a_3, a_1\, a_1, a_2\, a_2)^T \sim \underline{\mathbf{3}}$,
  $$(a_{\{2}\, a_{3\}}, a_{\{3}\, a_{1\}}, a_{\{1}\, a_{2\}})^T \sim \underline{\mathbf{3}}^*$$

T_7: Realization

Particle Content

Field	$\mathbf{16}_i$	S_i	H	Δ	χ_i
$SO(10)$	$\underline{\mathbf{16}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{10}}$	$\underline{\mathbf{16}}$	$\underline{\mathbf{1}}$
T_7	$\underline{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{i}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{3}}^{\star}$

Superpotential

$$\begin{aligned} W \supset\ & \alpha\, H\,(\underline{\mathbf{16}}_3\,\underline{\mathbf{16}}_3\,\chi_1 + \underline{\mathbf{16}}_1\,\underline{\mathbf{16}}_1\,\chi_2 + \underline{\mathbf{16}}_2\,\underline{\mathbf{16}}_2\,\chi_3)/\Lambda \\ & + \beta_1\,(\underline{\mathbf{16}}_1\,\chi_1 + \underline{\mathbf{16}}_2\,\chi_2 + \underline{\mathbf{16}}_3\,\chi_3)\,\Delta\,S_1/\Lambda \\ & + \beta_2\,(\underline{\mathbf{16}}_1\,\chi_1 + \omega\,\underline{\mathbf{16}}_2\,\chi_2 + \omega^2\,\underline{\mathbf{16}}_3\,\chi_3)\,\Delta\,S_2/\Lambda \\ & + \beta_3\,(\underline{\mathbf{16}}_1\,\chi_1 + \omega^2\,\underline{\mathbf{16}}_2\,\chi_2 + \omega\,\underline{\mathbf{16}}_3\,\chi_3)\,\Delta\,S_3/\Lambda \\ & + A\,S_1\,S_1 + B\,(S_2\,S_3 + S_3\,S_2) + \mathrm{h.c.} \end{aligned}$$

T_7: Lowest Order

$$m_D = \frac{\alpha \, \langle H \rangle}{\Lambda} \begin{pmatrix} \langle \chi_2 \rangle & 0 & 0 \\ 0 & \langle \chi_3 \rangle & 0 \\ 0 & 0 & \langle \chi_1 \rangle \end{pmatrix}$$

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$$M_{NS} = \frac{\langle \Delta \rangle_N}{\Lambda} \begin{pmatrix} \langle \chi_1 \rangle & 0 & 0 \\ 0 & \langle \chi_2 \rangle & 0 \\ 0 & 0 & \langle \chi_3 \rangle \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

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$$M_{SS} = \begin{pmatrix} A & 0 & 0 \\ \cdot & 0 & B \\ \cdot & \cdot & 0 \end{pmatrix}$$

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$$M_{SS} = \begin{pmatrix} A & 0 & 0 \\ \cdot & 0 & B \\ \cdot & \cdot & 0 \end{pmatrix}$$

$$m_\nu \approx \left(\frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N} \right)^2 D_\chi \begin{pmatrix} \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \\ \cdot & \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} \\ \cdot & \cdot & \tilde{A} + 2\tilde{B} \end{pmatrix} D_\chi$$

$$\tilde{A} \equiv \frac{A}{9\beta_1^2}, \quad \tilde{B} \equiv \frac{B}{9\beta_2\beta_3}, \quad D_\chi \equiv \text{diag}\left(\frac{\langle \chi_2 \rangle}{\langle \chi_1 \rangle}, \frac{\langle \chi_3 \rangle}{\langle \chi_2 \rangle}, \frac{\langle \chi_1 \rangle}{\langle \chi_3 \rangle} \right)$$

T_7: Lowest Order

$$m_D = \frac{\alpha \, \langle H \rangle \, \langle \chi_1 \rangle}{\Lambda} \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{NS} = \frac{\langle \Delta \rangle_N \, \langle \chi_1 \rangle}{\Lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

$$M_{SS} = \begin{pmatrix} A & 0 & 0 \\ \cdot & 0 & B \\ \cdot & \cdot & 0 \end{pmatrix}$$

$$m_\nu \approx \left(\frac{\alpha \, \langle H \rangle}{\langle \Delta \rangle_N \, \epsilon^2} \right)^2 \begin{pmatrix} (\tilde{A} + 2\tilde{B})\epsilon^{12} & (\tilde{A} - \tilde{B})\epsilon^6 & (\tilde{A} - \tilde{B})\epsilon^6 \\ \cdot & \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} \\ \cdot & \cdot & \tilde{A} + 2\tilde{B} \end{pmatrix}$$

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T_7: Higher-Dimensional Operators and Mass Scales

- Higher-dimensional operators up to $\frac{m_u}{\alpha} \sim \epsilon^4 \eta$, $\eta = \frac{\langle \chi_1 \rangle}{\Lambda} \sim 0.48$

$$\chi_1^n \xrightarrow{A} \mathrm{e}^{-\frac{2\pi i}{7} n} \chi_1^n \sim \mathcal{O}(1) \Rightarrow \text{Introduction of } Z_7$$

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- Observe: only one covariant

$$SS\chi^3/\Lambda^2 \sim \left(a\, S_1 S_1 + b\left(S_2 S_3 + S_3 S_2\right)\right) \chi_1 \chi_2 \chi_3 / \Lambda^2$$

Field	$\underline{\mathbf{16}}_i$	S_i	H	Δ	χ_i
T_7	$\underline{\mathbf{3}}$	$\underline{\mathbf{1}}_i$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{3}}^*$
Z_7	3	2	0	1	1

T_7: Solar Mixing Angle

$$m_\nu^{LS} \approx -\left[m_D\, {M_{NS}^{-1}}^T\, m_{\nu S}^T + (\dots)^T\right]$$

$m_{\nu S}$ originates from $\underline{\mathbf{16}}_i\, \Delta\, S_j\, \chi \Rightarrow m_\nu^{LS}$ diagonal

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$$m_{\nu S} = \frac{\langle \Delta' \rangle_\nu}{\Lambda} \begin{pmatrix} \langle\chi_1\rangle & 0 & 0 \\ 0 & \langle\chi_2\rangle & 0 \\ 0 & 0 & \langle\chi_3\rangle \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \beta_1' & 0 & 0 \\ 0 & \beta_2' & 0 \\ 0 & 0 & \beta_3' \end{pmatrix}$$

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Leading order

$$m_\nu \approx \left(\frac{\alpha\,\langle H\rangle}{\langle\Delta\rangle_N\,\epsilon^2}\right)^2\begin{pmatrix}-2X\sum_{i=1}^3\gamma_i\,\epsilon^6 & -X\sum_{i=1}^3\gamma_i\,\omega^{1-i} & -X\sum_{i=1}^3\gamma_i\,\omega^{i-1}\\ . & \tilde{A}+2\tilde{B} & \tilde{A}-\tilde{B}\\ . & . & \tilde{A}+2\tilde{B}\end{pmatrix}$$

$$X = \frac{\langle\Delta\rangle_N\,\langle\Delta'\rangle_\nu\,\langle\chi_1\rangle\,\epsilon^2}{3\,\alpha\,\langle H\rangle\,\Lambda}\,,\qquad \gamma_i = \beta'_i/\beta_i$$

T_7: Phenomenology

Dominant 2-3 block in m_ν preserved, θ_{12}, θ_{13} can be fitted:

$$\tan 2\theta_{12} \approx \frac{\sqrt{2}|(2\gamma_1 - \gamma_2 - \gamma_3)X|}{|2\tilde{A} + \tilde{B}|} , \quad \sin\theta_{13} \approx \frac{|(\gamma_2 - \gamma_3)\,X|}{\sqrt{6}\,|\tilde{B}|} , \quad \theta_{23} \approx \frac{\pi}{4}$$

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m_1 and m_2 especially changed:

$$m_1 \approx \left(\frac{\alpha\,\langle H\rangle}{\langle\Delta\rangle_N\,\epsilon^2}\right)^2 |2\,\tilde{A} + \tilde{B}| \left|\frac{\sin^2\theta_{12}}{\cos 2\theta_{12}}\right|$$

$$m_2 \approx \left(\frac{\alpha\,\langle H\rangle}{\langle\Delta\rangle_N\,\epsilon^2}\right)^2 |2\,\tilde{A} + \tilde{B}| \left|\frac{\cos^2\theta_{12}}{\cos 2\theta_{12}}\right|$$

$$m_3 \approx \left(\frac{\alpha\,\langle H\rangle}{\langle\Delta\rangle_N\,\epsilon^2}\right)^2 3|\tilde{B}|$$

$\Rightarrow \frac{m_1}{m_2} \approx \tan^2\theta_{12}$

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$$\Rightarrow \frac{m_1}{m_2} \approx \tan^2\theta_{12}$$

$$|2\tilde{A} + \tilde{B}| \approx 1.13 \cdot 10^9\,\mathrm{GeV}$$
$$|\tilde{B}| \approx 3.38 \cdot 10^9\,\mathrm{GeV}$$
$$|2\gamma_1 - \gamma_2 - \gamma_3| \approx 0.0833$$
$$X \approx 2.25 \cdot 10^{10}\,\mathrm{GeV}$$
$$\langle\Delta\rangle_N = 10^{15}\,\mathrm{GeV}$$
$$\langle\Delta'\rangle_\nu = 10\,\mathrm{GeV}$$
$$\Rightarrow m_1 \approx 0.00424\,\mathrm{eV}$$
$$M_i \sim 10^{9-15}\,\mathrm{GeV}$$

T_7: Cabibbo Angle and Flavon Potential

Cabibbo Angle

By introduction of $\underline{\mathbf{16}}_H, \underline{\mathbf{16}}'_H \sim (\underline{\mathbf{1}}_{T_7}, 6_{Z_7})$

$$\frac{1}{M}\left(\underline{\mathbf{16}}_i\underline{\mathbf{16}}_j\underline{\mathbf{16}}_H\underline{\mathbf{16}}'_H\right)\left(\frac{\chi}{\Lambda}\right)^3$$

contributes to down type and charged lepton mass matrix

$$m_{down} \approx \langle\underline{\mathbf{16}}_H\rangle_\nu \left(\frac{\langle\underline{\mathbf{16}}'_H\rangle_N}{M}\right)\begin{pmatrix} \mathcal{O}(\epsilon^4\eta^3) & \mathcal{O}(\eta^3) & \mathcal{O}(\epsilon^6\eta^3) \\ . & \mathcal{O}(\epsilon^4\eta^3) & \mathcal{O}(\epsilon^2\eta^3) \\ . & . & 0 \end{pmatrix}$$

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Flavon Potential

Leading order can be explained with driving field $\varphi \sim (\underline{\mathbf{3}}^*_{T_7}, 5_{Z_7})$:

$$W_\chi = \kappa\left(\varphi_1\chi_2\chi_3 + \varphi_2\chi_3\chi_1 + \varphi_3\chi_1\chi_2\right)$$

$\Rightarrow$ further study required

$\Sigma(81)$: Realization

Field	$\underline{\mathbf{16}}_i$	S_i	H	Δ	χ_i
$SO(10)$	$\underline{\mathbf{16}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{10}}$	$\overline{\underline{\mathbf{16}}}$	$\underline{\mathbf{1}}$
$\Sigma(81)$	$\underline{\mathbf{3}}_1$	$\underline{\mathbf{1}}_{\mathbf{i}}$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{3}}_2 \simeq \underline{\mathbf{3}}_1^\star$

$$m_\nu \approx \left(\frac{\alpha \langle H\rangle}{\langle\Delta\rangle_N}\right)^2 \begin{pmatrix} \tilde{A}+2\tilde{B} & \tilde{A}-\tilde{B} & \tilde{A}-\tilde{B} \\ \cdot & \tilde{A}+2\tilde{B} & \tilde{A}-\tilde{B} \\ \cdot & \cdot & \tilde{A}+2\tilde{B} \end{pmatrix}$$

$$\tilde{A} \equiv A/(9\beta_1^2), \quad \tilde{B} \equiv B/(9\beta_2\beta_3)$$

$$m_D = \frac{\alpha \langle H\rangle}{\Lambda} \begin{pmatrix} \langle\chi_1\rangle^* & 0 & 0 \\ 0 & \langle\chi_2\rangle^* & 0 \\ 0 & 0 & \langle\chi_3\rangle^* \end{pmatrix} \qquad M_{SS} = \begin{pmatrix} A & 0 & 0 \\ \cdot & 0 & B \\ \cdot & \cdot & 0 \end{pmatrix}$$

$$M_{NS} = \frac{\langle\Delta\rangle_N}{\Lambda} \begin{pmatrix} \langle\chi_1\rangle & 0 & 0 \\ 0 & \langle\chi_2\rangle & 0 \\ 0 & 0 & \langle\chi_3\rangle \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

$\Sigma(81)$: Phenomenology

Charged Fermions

- Quark mass hierarchy $\langle\chi_1\rangle^* : \langle\chi_2\rangle^* : \langle\chi_3\rangle^* = \epsilon^4 : \epsilon^2 : 1$, $\epsilon \sim 0.05$
- Zero mixing in quark sector
- $m_t \Rightarrow \langle\chi_3\rangle^* \sim \Lambda \Rightarrow$ higher-dimensional operators are relevant

Neutrinos

- Dirac mass hierarchy exactly drops out

$$m_\nu \approx \left(\frac{\alpha\ \langle H\rangle}{\langle\Delta\rangle_N}\right)^2 \begin{pmatrix} \tilde{A}+2\tilde{B} & \tilde{A}-\tilde{B} & \tilde{A}-\tilde{B} \\ \cdot & \tilde{A}+2\tilde{B} & \tilde{A}-\tilde{B} \\ \cdot & \cdot & \tilde{A}+2\tilde{B} \end{pmatrix}$$

- Neutrino mass matrix diagonalized by tri-bimaximal mixing matrix
- $m_2 = 3\left|\frac{\alpha\langle H\rangle}{\langle\Delta\rangle_N}\right|^2 \left|\tilde{A}\right|$, $m_{1,3} = 3\left|\frac{\alpha\langle H\rangle}{\langle\Delta\rangle_N}\right|^2 \left|\tilde{B}\right|$
  $\rightarrow$ corrections from higher-dimensional operators lift degeneracy

Outline

Summary & Outlook

Cancellation Mechanism

- Different hierarchies in charged and neutral fermion masses within SO(10)

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- Further investigation of flavon potential needed
- Flavor symmetry with complete cancellation and no degeneracies
- Implementation in complete model

Thank you for your attention.

$\Sigma(81)$: Group Theory

- $\Sigma(81) \subset \mathrm{U}(3)$: order 81
- Irreducible representations: $\underline{\mathbf{1}}_{\mathbf{i}}$, $i = 1, \ldots, 9$ and $\underline{\mathbf{3}}_{\mathbf{i}}$, $i = 1, \ldots, 8$

Rep.	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{5}}$	$\underline{\mathbf{1}}_{\mathbf{6}}$	$\underline{\mathbf{3}}_{\mathbf{1}}$	$\underline{\mathbf{3}}_{\mathbf{3}}$	$\underline{\mathbf{3}}_{\mathbf{5}}$	$\underline{\mathbf{3}}_{\mathbf{7}}$
Rep.*	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{7}}$	$\underline{\mathbf{1}}_{\mathbf{8}}$	$\underline{\mathbf{1}}_{\mathbf{9}}$	$\underline{\mathbf{3}}_{\mathbf{2}}$	$\underline{\mathbf{3}}_{\mathbf{4}}$	$\underline{\mathbf{3}}_{\mathbf{6}}$	$\underline{\mathbf{3}}_{\mathbf{8}}$

- Kronecker products:
  - like in T_7:

$$\underline{\mathbf{1}}_{\mathbf{i}} \otimes \underline{\mathbf{1}}_{\mathbf{j}} = \underline{\mathbf{1}}_{\mathbf{i+j} \bmod \mathbf{3}} \,, \qquad i, j = 1, 2, 3$$

$$\underline{\mathbf{3}}_{\mathbf{i}} \otimes \underline{\mathbf{1}}_{\mathbf{j}} = \underline{\mathbf{3}}_{\mathbf{i}} \,, \quad i = 1, 2 \,; j = 1, 2, 3$$

  - $\{\underline{\mathbf{3}}_{\mathbf{1}} \otimes \underline{\mathbf{3}}_{\mathbf{1}}\} = \underline{\mathbf{3}}_{\mathbf{2}} \oplus \underline{\mathbf{3}}_{\mathbf{4}}$:
    $(a_1,\, a_2,\, a_3)^T \sim \underline{\mathbf{3}}_{\mathbf{1}}$: $(a_1\, a_1, a_2\, a_2, a_3\, a_3)^T \sim \underline{\mathbf{3}}_{\mathbf{2}}$ with $\omega = e^{i2\pi/3}$
  - $\underline{\mathbf{3}}_{\mathbf{1}} \otimes \underline{\mathbf{3}}_{\mathbf{2}} = \underline{\mathbf{1}}_{\mathbf{1}} \oplus \underline{\mathbf{1}}_{\mathbf{2}} \oplus \underline{\mathbf{1}}_{\mathbf{3}} \oplus \underline{\mathbf{3}}_{\mathbf{7}} \oplus \underline{\mathbf{3}}_{\mathbf{8}}$:
    $(a_1, a_2, a_3)^T \sim \underline{\mathbf{3}}_{\mathbf{1}}$, $(b_1, b_2, b_3)^T \sim \underline{\mathbf{3}}_{\mathbf{2}}$: $(a_1\, b_1, a_2\, b_2, a_3\, b_3)^T \sim \underline{\mathbf{1}}_{\mathbf{1}}$

$\Sigma(81)$: Realization

Particle Content

Field	$\underline{\mathbf{16}}_i$	S_i	H	Δ	χ_i
$SO(10)$	$\underline{\mathbf{16}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{10}}$	$\overline{\underline{\mathbf{16}}}$	$\underline{\mathbf{1}}$
$\Sigma(81)$	$\underline{\mathbf{3}}_1$	$\underline{\mathbf{1}}_\mathbf{i}$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{3}}_2 \cong \underline{\mathbf{3}}_1^\star$

Lagrangian

$$
\begin{aligned}
\mathscr{L} \supset\ & \alpha\, H\,(\underline{\mathbf{16}}_3\,\underline{\mathbf{16}}_3\,\chi_1^* + \underline{\mathbf{16}}_1\,\underline{\mathbf{16}}_1\,\chi_2^* + \underline{\mathbf{16}}_2\,\underline{\mathbf{16}}_2\,\chi_3^*)/\Lambda \\
& + \beta_1\,(\underline{\mathbf{16}}_1\,\chi_1 + \underline{\mathbf{16}}_2\,\chi_2 + \underline{\mathbf{16}}_3\,\chi_3)\,\Delta\,S_1/\Lambda \\
& + \beta_2\,(\underline{\mathbf{16}}_1\,\chi_1 + \omega\,\underline{\mathbf{16}}_2\,\chi_2 + \omega^2\,\underline{\mathbf{16}}_3\,\chi_3)\,\Delta\,S_2/\Lambda \\
& + \beta_3\,(\underline{\mathbf{16}}_1\,\chi_1 + \omega^2\,\underline{\mathbf{16}}_2\,\chi_2 + \omega\,\underline{\mathbf{16}}_3\,\chi_3)\,\Delta\,S_3/\Lambda \\
& + A\,S_1\,S_1 + B\,(S_2\,S_3 + S_3\,S_2) + \mathrm{h.c.}
\end{aligned}
$$

$\Sigma(81)$: Lowest Order

$$m_D = \frac{\alpha \langle H \rangle}{\Lambda} \begin{pmatrix} \langle \chi_1 \rangle^* & 0 & 0 \\ 0 & \langle \chi_2 \rangle^* & 0 \\ 0 & 0 & \langle \chi_3 \rangle^* \end{pmatrix}$$

$$M_{NS} = \frac{\langle \Delta \rangle_N}{\Lambda} \begin{pmatrix} \langle \chi_1 \rangle & 0 & 0 \\ 0 & \langle \chi_2 \rangle & 0 \\ 0 & 0 & \langle \chi_3 \rangle \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

$$M_{SS} = \begin{pmatrix} A & 0 & 0 \\ \cdot & 0 & B \\ \cdot & \cdot & 0 \end{pmatrix}$$

$$m_\nu \approx \left(\frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N} \right)^2 \begin{pmatrix} \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \\ \cdot & \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} \\ \cdot & \cdot & \tilde{A} + 2\tilde{B} \end{pmatrix}$$

$$\tilde{A} \equiv A/(9\beta_1^2), \quad \tilde{B} \equiv B/(9\beta_2\beta_3)$$

$\Sigma(81)$: Higher-Dimensional Operators

Order in ϵ	Structure	Representation
$\mathcal{O}(1)$	$\chi_3^m (\chi_3^\star)^{n-m}$ $(m = 0, ..., n)$	$\underline{\mathbf{1}}_{\mathbf{1},\mathbf{2},\mathbf{3}}$ for $(2\,m - n) \mod 3 = 0$ 3^{rd} comp. of $\underline{\mathbf{3}}_\mathbf{1}$ for $(2\,m - n) \mod 3 = 1$ 3^{rd} comp. of $\underline{\mathbf{3}}_\mathbf{2}$ for $(2\,m - n) \mod 3 = 2$
$\mathcal{O}(\epsilon^2)$	...	...

- Structure of operators calculable to arbitrary order
- All higher-dimensional operators suppressed compared to LO.
- Vanishing entries are filled.

$\Sigma(81)$: Flavon Potential

In polar coordinates $\chi_i = X_i e^{\mathrm{i}\xi_i}$

$$V_\chi(X_j, \xi_j) = M_\chi^2 \sum_i X_i^2 + \lambda_1 \sum_i X_i^4 + \lambda_2 \sum_{i\neq k} X_i^2 X_k^2 + 2\kappa \sum_i X_i^3 \cos(\alpha + 3\xi_i)$$

Minimization:

$$\frac{\partial V_\chi}{\partial X_1} = 2X_1 \left(M_\chi^2 + 2\lambda_1 X_1^2 + \lambda_2 X_2^2 + \lambda_2 X_3^2 + 3\kappa X_1 \cos(\alpha + 3\xi_1)\right) \stackrel{!}{=} 0$$

$$\frac{\partial V_\chi}{\partial \xi_1} = -6\kappa X_1^3 \sin(\alpha + 3\xi_1) \stackrel{!}{=} 0 \qquad \text{and cyclic}$$

Minimum

$$\langle X_1 \rangle = \langle X_2 \rangle = 0 \,, \quad \langle X_3 \rangle = \frac{3\kappa + \sqrt{9\kappa^2 - 8M_\chi^2 \lambda_1}}{4\lambda_1} \,, \quad \langle \xi_3 \rangle = -\frac{\alpha \pm \pi}{3}$$

possible in a certain region of parameter space $(M_\chi^2, \lambda_1, \lambda_2, \kappa, \alpha)$.