Lepton Mixing and Discrete Flavor Symmetries

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Outline

- Observations in the lepton sector
- General comments on the flavor symmetry G_F
- Predicting TBM with the tetrahedral group A_4

  (Altarelli/Feruglio ('05), de Medeiros Varzielas et al. ('05), He et al. ('06))

  S_4 - more appropriate ?

  (Lam ('06, '07, '08))
- Predicting $\mu\tau$ symmetry with the dihedral group D_4

  (Grimus/Lavoura ('03), Adulpravitchai et al. ('08), Ishimori et al. ('08))
- Conclusions & Outlook

Observations in the Lepton Sector

- The mixing pattern is very peculiar ***(Maltoni et al. ('04), Schwetz et al. ('08))***

$$\sin^2(\theta_{12}) = 0.304^{+0.046}_{-0.034}\,, \quad \sin^2(\theta_{23}) = 0.50^{+0.13}_{-0.11} \quad \text{and} \quad \sin^2(\theta_{13}) \leq 0.040$$

$$\theta_{12} = (33.5^{+2.8}_{-2.2})^\circ\,, \quad \theta_{23} = (45.0^{+7.5}_{-6.4})^\circ \quad \text{and} \quad \theta_{13} \leq 11.5^\circ \qquad (2\,\sigma)$$

compare to quark sector $\theta^q_{12} \approx 13.0^\circ\,, \quad \theta^q_{23} \approx 2.4^\circ$ and $\theta^q_{13} \approx 0.21^\circ$

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- Special mixing patterns
  - $\mu\tau$ symmetry ***(Fukuyama/Nishiura ('97), Mohapatra/Nussinov ('99), Lam ('01), ...)***

$$\sin^2(\theta_{23}) = \tfrac{1}{2} \; , \quad \sin^2(\theta_{13}) = 0$$

$$\Rightarrow U_{MNS} = \begin{pmatrix} \cos(\theta_{12}) & \sin(\theta_{12}) & 0 \\ -\frac{\sin(\theta_{12})}{\sqrt{2}} & \frac{\cos(\theta_{12})}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin(\theta_{12})}{\sqrt{2}} & -\frac{\cos(\theta_{12})}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

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- Special mixing patterns
  - Tri-bimaximal mixing (TBM) ***(Harrison et al. ('02), Xing ('02))***

$$\sin^2(\theta_{12}) = \frac{1}{3}\;, \quad \sin^2(\theta_{23}) = \frac{1}{2}\;, \quad \sin^2(\theta_{13}) = 0$$

$$\Rightarrow U_{MNS} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Observations in the Lepton Sector

- Mild hierarchy among light neutrino masses
  - Two mass squared differences Δm^2_{21} and $|\Delta m^2_{31}|$ are known ($2\,\sigma$)

    (Maltoni et al. ('04), Schwetz et al. ('08))

$$\Delta m^2_{21} = \left(7.65^{+0.46}_{-0.40}\right) \cdot 10^{-5}\ \mathrm{eV}^2 \quad \text{and} \quad |\Delta m^2_{31}| = \left(2.40^{+0.24}_{-0.22}\right) \cdot 10^{-3}\ \mathrm{eV}^2$$

  - Cosmological data give upper bound on m_0 *(Fogli et al. ('08))*

$$\sum_i m_i \lesssim (0.19...1.19)\ \mathrm{eV} \qquad (2\,\sigma)$$

  - The bounds on m_β and $|m_{ee}|$ also constrain m_0

    (Kraus et al. ('04), Lobashev ('03); Klapdor et al. ('01), Aalseth et al. ('02), Arnaboldi et al. ('05), Arnold et al. ('05))

$$m_\beta \leq 2.2\,\mathrm{eV} \quad \text{and} \quad |m_{ee}| \leq 0.9\,\mathrm{eV}$$

- Normal (NH) & inverted hierarchy (IH) still allowed
- Charged lepton masses are strongly hierarchical,

$$m_e : m_\mu : m_\tau \approx \lambda^{4 \div 5} : \lambda^2 : 1 \quad \text{where } \lambda \approx \theta_C \approx 0.22$$

Effect of Flavor Symmetry G_F

- Yukawa couplings in the SM

$$y^{\nu}_{ij}\, L_i^T\, H^c\, \nu^c_j \quad \text{or} \quad y^{l}_{ij}\, L_i^T\, H\, e^c_j$$

with $y^{\nu,l}_{ij} \in \mathbb{C}$

- Enforce invariance under G_F
  $\hookrightarrow$ Constraints on $y^{\nu,l}_{ij}$
  $\hookrightarrow$ Extension of scalar sector needed

$H \longrightarrow H_k$	or	$H \longrightarrow H\, \phi_k$
multi-Higgs doublets	or	flavon fields
$y^{l}_{ij,k}\, L_i^T\, H_k\, e^c_j$	or	$y^{l}_{ij,k}\, L_i^T\, H\, e^c_j \left(\frac{\phi_k}{(M,\Lambda)}\right)$
renormalizable couplings	or	in general non-renormalizable

Choice of Flavor Symmetry G_F

The symmetry G_F could ...

- ... be abelian or non-abelian

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- ... be broken at low or high energies (low: electroweak scale)
  (high: seesaw/GUT scale)

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  (high: seesaw/GUT scale)

Its maximal possible size depends on the gauge group,
e.g. in the SM without ν^c: $G_F \subset U(3)^5$,
in $SO(10)$: $G_F \subset U(3)$.

Possible Symmetries

- **Continuous Groups**

$$SU(2),\ U(2),\ SO(3),\ SU(3),\ U(3)$$

- **Discrete Groups**
  - Permutation symmetries, S_N and A_N with $N \in \mathbb{N}$
  - Dihedral symmetries, D_n and D'_n with $n \in \mathbb{N}$
  - Further double-valued groups: T', O', I', ...
  - Subgroups of $SU(3)$, series of $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$ groups with $n \in \mathbb{N}$, as well as finite number of Σ groups
  - Additional groups such as subgroups of the mentioned groups, e.g. $T_7 \cong Z_7 \rtimes Z_3 \subset \Delta(147)$, subgroups of $U(3)$, e.g. $\Sigma(81)$

NB: Several isomorphisms among the groups, e.g. $S_3 \cong D_3 \cong \Delta(6)$

TBM from Non-Trivial A_4 Breaking

(Altarelli/Feruglio ('05), de Medeiros Varzielas et al. ('0 He et al. ('06))

- Family symmetry $G_F = A_4$ is spontaneously broken at high energies
- Low energy effective theory: MSSM
- Breaking of G_F is induced by VEVs of flavon fields which are singlets under the SM gauge group
- (MS)SM fermions transform non-trivially under G_F
- MSSM Higgs doublets $h_{u,d}$ are singlets under G_F
- Elaborate construction of scalar sector needed to ensure vacuum alignment

Particle Content

- (MS)SM fermions transform under G_F

$$L \equiv \{L_i\} \sim 3\,, \quad e^c_L \sim 1\,, \quad \mu^c_L \sim 1''\,, \quad \tau^c_L \sim 1'$$

- MSSM Higgs doublets $h_{u,d}$ are singlets under G_F

$$h_{u,d} \sim 1$$

- Flavon fields

$$\varphi_S \sim 3\,, \quad \xi, \tilde{\xi} \sim 1 \quad \text{and} \quad \varphi_T \sim 3$$

- Mass terms

$$L_i^T\, h_d\, e^c_L \left(\frac{\phi_k}{\Lambda}\right)$$

$$\frac{1}{\Lambda}\, L_i^T\, h_u\, L_j^T\, h_u \left(\frac{\phi_k}{\Lambda}\right)$$

Particle Content

		LEPTONS				FLAVONS			
Field	L	e^c_L	μ^c_L	τ^c_L	φ_T	φ_S	$\xi, \tilde{\xi}$	θ	
A_4	3	1	$1''$	$1'$	3	3	1	1	family dependent
Z_3	ω	ω^2	ω^2	ω^2	1	ω	ω	1	family independent
$U(1)_{FN}$	0	2	1	0	0	0	0	-1	family dependent

Additionally needed

- Z_3 symmetry to separate charged lepton and neutrino sector

$$\{\varphi_T\} \rightarrow \mathcal{M}_l \quad \text{and} \quad \left\{\varphi_S, \xi, \tilde{\xi}\right\} \rightarrow M_\nu$$

- $U(1)_{FN}$ for hierarchy $m_e \ll m_\mu \ll m_\tau$

Group Theory of A_4

- The group A_4 is the symmetry group of a regular tetrahedron, group of even permutations of four objects
- Order of the group is 12
- Irred reps are 1, $1'$, $1''$ and 3
- Kronecker products

$$1 \times \mu = \mu \quad \forall \quad \mu\ ,$$
$$1' \times 1' = 1''\ , \quad 1'' \times 1'' = 1'\ , \quad 1' \times 1'' = 1$$
$$1 \times 3 = 1' \times 3 = 1'' \times 3 = 3$$
$$3 \times 3 = 1 + 1' + 1'' + 3 + 3$$

Group Theory of A_4

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- Order of the group is 12
- Irred reps are 1, $1'$, $1''$ and 3
- Generator relations for generators S and T

$$S^2 = \mathbb{1}\,, \quad T^3 = \mathbb{1}\,, \quad (ST)^3 = \mathbb{1}$$

rep.	S	T
1	1	1
$1'$	1	ω^2
$1''$	1	ω
3	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$

$(\omega = e^{\frac{2\pi i}{3}})$

(Altarelli/Feruglio ('05), 2nd paper)

Superpotential

$$w = w_l + w_d$$

where

$$\begin{aligned}
w_l &= \frac{y_e}{\Lambda^3} (\varphi_T L) e^c_L \theta^2 h_d + \frac{y_\mu}{\Lambda^2} (\varphi_T L)' \mu^c_L \theta h_d + \frac{y_\tau}{\Lambda} (\varphi_T L)'' \tau^c_L h_d \\
&\quad + \frac{1}{\Lambda^2} (x_a \xi + \tilde{x}_a \tilde{\xi}) (L L) h_u h_u + \frac{x_b}{\Lambda^2} (\varphi_S L L) h_u h_u \\
&= \frac{y_e}{\Lambda^3} (\varphi_{T1} L_1 + \varphi_{T2} L_3 + \varphi_{T3} L_2) e^c_L \theta^2 h_d \\
&\quad + \frac{y_\mu}{\Lambda^2} (\varphi_{T1} L_2 + \varphi_{T2} L_1 + \varphi_{T3} L_3) \mu^c_L \theta h_d \\
&\quad + \frac{y_\tau}{\Lambda} (\varphi_{T1} L_3 + \varphi_{T2} L_2 + \varphi_{T3} L_1) \tau^c_L h_d \\
&+ \frac{1}{\Lambda^2} (x_a \xi + \tilde{x}_a \tilde{\xi}) (L_1 L_1 + L_2 L_3 + L_3 L_2) h_u h_u \\
&+ \frac{x_b}{3\Lambda^2} [\, \varphi_{S1} (2L_1L_1 - L_2L_3 - L_3L_2) + \varphi_{S2} (2L_2L_2 - L_1L_3 - L_3L_1) \\
&\qquad + \varphi_{S3} (2L_3L_3 - L_1L_2 - L_2L_1) \,] \, h_u h_u
\end{aligned}$$

Fermion Masses at LO

- Assume vacuum

$$G_S \quad : \qquad \langle \varphi_S \rangle = (v_S, v_S, v_S)\ , \quad \langle \xi \rangle = u\ , \quad \langle \tilde{\xi} \rangle = 0\ ,$$

$$G_T \quad : \qquad \langle \varphi_T \rangle = (v_T, 0, 0)\ .$$

- Charged lepton sector

$$\mathcal{M}_l = \frac{v_T}{\Lambda} v_d \, \text{diag} \left(y_e \frac{\langle \theta \rangle^2}{\Lambda^2}, y_\mu \frac{\langle \theta \rangle}{\Lambda}, y_\tau \right)$$

- Neutrino sector

$$M_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a + 2b/3 & -b/3 & -b/3 \\ -b/3 & 2b/3 & a - b/3 \\ -b/3 & a - b/3 & 2b/3 \end{pmatrix} \Rightarrow \boxed{\text{TB mixing!}}$$

with masses $\frac{v_u^2}{\Lambda}$ diag$(a+b, a, -a+b)$ and $a = x_a \frac{u}{\Lambda}$, $b = x_b \frac{v_S}{\Lambda}$

Non-Trivial A_4 Breaking

- Observation
  - $\langle \varphi_S \rangle = (v_S, v_S, v_S)$, $\langle \xi \rangle = u$ breaks $A_4 \to G_S \cong Z_2$ in ν sector
  - $\langle \varphi_T \rangle = (v_T, 0, 0)$ breaks $A_4 \to G_T \cong Z_3$ in l sector
  - A_4 *completely* broken in the whole theory

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  - A_4 *completely* broken in the whole theory

In detail: Z_2 symmetry is generated by the element S of A_4

Since for 1, $1'$ and $1''$ $S = 1$, VEV $\langle\xi\rangle$ preserves S.

For $\langle\varphi_S\rangle$ the vector has to fulfill

$$\langle\varphi_S\rangle \propto v_{+1} \quad \text{with} \quad S\, v_{+1} = +1\, v_{+1}$$

i.e. $$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Non-Trivial A_4 Breaking

- Observation
  - $\langle \varphi_S \rangle = (v_S, v_S, v_S)$, $\langle \xi \rangle = u$ breaks $A_4 \to G_S \cong Z_2$ in ν sector
  - $\langle \varphi_T \rangle = (v_T, 0, 0)$ breaks $A_4 \to G_T \cong Z_3$ in l sector
  - A_4 *completely* broken in the whole theory

In detail: Z_3 symmetry is generated by the element T of A_4

For $\langle \varphi_T \rangle$ the vector has to fulfill

$$\langle \varphi_T \rangle \propto v_{+1} \quad \text{with} \quad T\, v_{+1} = +1\, v_{+1}$$

i.e. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Non-Trivial A_4 Breaking

- Observation
  - $\langle\varphi_S\rangle = (v_S, v_S, v_S)$, $\langle\xi\rangle = u$ breaks $A_4 \rightarrow G_S \cong Z_2$ in ν sector
  - $\langle\varphi_T\rangle = (v_T, 0, 0)$ breaks $A_4 \rightarrow G_T \cong Z_3$ in l sector
  - A_4 *completely* broken in the whole theory

A_4	D_2	Z_3	Z_2
1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$1'$	$\mathbf{1}$		$\mathbf{1}$
$1''$	$\mathbf{1}$		$\mathbf{1}$
3		$\mathbf{1}$ $+ \ldots$	$\mathbf{1}$ $+ \ldots$

with $\mathbf{1}$ being a total singlet of the subgroup

Non-Trivial A_4 Breaking

- Observation
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  - $\langle \varphi_T \rangle = (v_T, 0, 0)$ breaks $A_4 \to G_T \cong Z_3$ in l sector
  - A_4 *completely* broken in the whole theory

Mismatch of different subgroups generates non-trivial (large) mixing and even more predicts exact values of mixing angles (TB mixing) (independent of choice of parameters, apart from order of eigenvalues)

$$\Downarrow$$

Interpretation of Mixing

Is S_4 more appropriate for TBM? (Lam ('06,'07,'08))

- TBM only results from A_4, if two flavons ξ', ξ'' transforming as $1'$ and $1''$ are not present in the neutrino sector
- However: $\langle\xi'\rangle \neq 0$, $\langle\xi''\rangle \neq 0$ leave $Z_2 \subset A_4$ invariant
- Matrix M_ν

$$M_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a + 2b/3 & -b/3 & -b/3 \\ -b/3 & 2b/3 & a - b/3 \\ -b/3 & a - b/3 & 2b/3 \end{pmatrix}$$

is not only invariant under S

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \;, \quad \text{i.e.} \quad S^T M_\nu S = M_\nu$$

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but also under P_{23}

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;, \quad \text{i.e.} \quad P_{23}^T \, M_\nu \, P_{23} = M_\nu$$

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- However: $\langle \xi' \rangle \neq 0$, $\langle \xi'' \rangle \neq 0$ leave $Z_2 \subset A_4$ invariant
- Matrix M_ν is invariant under S and P_{23}

$$\Downarrow$$

M_ν is invariant under $Z_2 \times Z_2$

But only S is an element of A_4 and not P_{23}

$$\Downarrow$$

The actual symmetry must be larger: S_4

- **Caveat**: Whether the charged lepton masses can still be *naturally* generated by a $U(1)_{FN}$ is not clear.
  (possible, if no subgroup conserved in charged lepton sector ***(Bazzocchi et al. ('09))***)

Miscellaneous Comments

- Simple extension to quark sector is not successful ***(Altarelli/Feruglio ('05))***
main problem: θ_C turns out to be too small, since $V_{CKM} = \mathbb{1}$ at LO
(for alternatives, see: ***Ma ('02), He et al. ('06), Bazzocchi et al. ('07)***)
- Successful extension to quark sector possible via extending the family symmetry from A_4 to its double-covering T' ***(Feruglio et al. ('07))***
however: θ_C is a bit too small without tuning
- Embedding into $SU(5)$ GUT (in extra dimensions) is possible
(Altarelli et al. ('08))
however: up quark mass matrix is then only constrained by abelian family symmetries ($Z_3 \subset A_4$, $U(1)_{FN}$) and still θ_C is slightly tuned
- Several studies are concerned with other phenomenological imprints of the model, such as leptogenesis ***(Adhikary/Ghosal ('08), Jenkins/Manohar ('08))*** and lepton flavor violation and dipole moments ***(Feruglio et al. ('08))***

$\mu\tau$ Symmetry from Non-Trivial D_4 Breaking *(Grimus/Lavoura ('03))*

- Flavor symmetry is D_4 with additional symmetry $Z_2^{(aux)}$
- Leptons

$$D_e \sim (\underline{\mathbf{1}}_{\mathbf{+\,+}}, +)\ , \quad (D_\mu, D_\tau)^t \sim (\underline{\mathbf{2}}, +)\ ,$$
$$e_R \sim (\underline{\mathbf{1}}_{\mathbf{+\,+}}, -)\ , \quad (\mu_R, \tau_R)^t \sim (\underline{\mathbf{2}}, +)\ ,$$
$$\nu_{e\,R} \sim (\underline{\mathbf{1}}_{\mathbf{+\,+}}, -)\ , \quad (\nu_{\mu\,R}, \nu_{\tau\,R})^t \sim (\underline{\mathbf{2}}, -)$$

- Scalars ((ϕ_i are $SU(2)_L$ doublets, χ_i gauge singlets)

$$\phi_1 \sim (\underline{\mathbf{1}}_{\mathbf{+\,+}}, -)\ ,$$
$$\phi_2 \sim (\underline{\mathbf{1}}_{\mathbf{+\,+}}, +)\ , \quad \phi_3 \sim (\underline{\mathbf{1}}_{\mathbf{+\,-}}, +)\ ,$$
$$(\chi_1, \chi_2)^t \sim (\underline{\mathbf{2}}, +)$$

Group Theory of D_4

- D_4 belongs to the dihedral groups and is the symmetry group of a square
- Its order is 8, i.e. it has eight distinct elements
- It has 5 irred. reps., $\underline{\mathbf{1}}_{++}$, $\underline{\mathbf{1}}_{+-}$, $\underline{\mathbf{1}}_{-+}$, $\underline{\mathbf{1}}_{--}$, $\underline{\mathbf{2}}$
- Kronecker products

$$\underline{\mathbf{1}}_{++} \times \mu = \mu \quad \forall \quad \mu \; ,$$
$$\underline{\mathbf{1}}_{+-} \times \underline{\mathbf{1}}_{+-} = \underline{\mathbf{1}}_{-+} \times \underline{\mathbf{1}}_{-+} = \underline{\mathbf{1}}_{--} \times \underline{\mathbf{1}}_{--} = \underline{\mathbf{1}}_{++} \; ,$$
$$\underline{\mathbf{1}}_{+-} \times \underline{\mathbf{1}}_{-+} = \underline{\mathbf{1}}_{--} \; , \quad \underline{\mathbf{1}}_{+-} \times \underline{\mathbf{1}}_{--} = \underline{\mathbf{1}}_{-+} \; , \quad \underline{\mathbf{1}}_{-+} \times \underline{\mathbf{1}}_{--} = \underline{\mathbf{1}}_{+-} \; ,$$
$$\underline{\mathbf{1}}_{+-} \times \underline{\mathbf{2}} = \underline{\mathbf{1}}_{-+} \times \underline{\mathbf{2}} = \underline{\mathbf{1}}_{--} \times \underline{\mathbf{2}} = \underline{\mathbf{2}} \; ,$$
$$\underline{\mathbf{2}} \times \underline{\mathbf{2}} = \underline{\mathbf{1}}_{++} + \underline{\mathbf{1}}_{+-} + \underline{\mathbf{1}}_{-+} + \underline{\mathbf{1}}_{--}$$

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- It has 5 irred. reps., $\underline{\mathbf{1}}_{\mathbf{++}}$, $\underline{\mathbf{1}}_{\mathbf{+-}}$, $\underline{\mathbf{1}}_{\mathbf{-+}}$, $\underline{\mathbf{1}}_{\mathbf{--}}$, $\underline{\mathbf{2}}$
- Generator relations of D_4

$$g^2 = \mathbb{1}\ , \quad h^2 = \mathbb{1}\ , \quad (g\,h)^4 = \mathbb{1}$$

- Generators

$$g = +1 \quad , \quad h = +1 \quad \text{... for } \underline{\mathbf{1}}_{++}$$

$$g = +1 \quad , \quad h = -1 \quad \text{... for } \underline{\mathbf{1}}_{+-}$$

$$g = -1 \quad , \quad h = +1 \quad \text{... for } \underline{\mathbf{1}}_{-+}$$

$$g = -1 \quad , \quad h = -1 \quad \text{... for } \underline{\mathbf{1}}_{--}$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{... for } \underline{\mathbf{2}}$$

$\mu\tau$ Symmetry from Non-Trivial D_4 Breaking

- Yukawa couplings

$$\begin{aligned}\mathcal{L}_Y &= y_1\,\overline{D}_e\,\nu_{e\,R}\,\phi_1^c + y_2\,(\overline{D}_\mu\,\nu_{\mu\,R} + \overline{D}_\tau\,\nu_{\tau\,R})\,\phi_1^c \\ &\quad + y_3\,\overline{D}_e\,e_R\,\phi_1 + y_4\,(\overline{D}_\mu\,\mu_R + \overline{D}_\tau\,\tau_R)\,\phi_2 + y_5\,(\overline{D}_\mu\,\mu_R - \overline{D}_\tau\,\tau_R)\,\phi_3 + \text{h.c.} \\ \mathcal{L}_{\nu_R} &= M\,\nu_{e\,R}\,\nu_{e\,R} + M'\,(\nu_{\mu\,R}\,\nu_{\mu\,R} + \nu_{\tau\,R}\,\nu_{\tau\,R}) \\ &\quad + y_\chi\,(\nu_{e\,R}\,\nu_{\mu\,R}\,\chi_1 + \nu_{e\,R}\,\nu_{\tau\,R}\,\chi_2) + y_\chi\,(\nu_{\mu\,R}\,\nu_{e\,R}\,\chi_1 + \nu_{\tau\,R}\,\nu_{e\,R}\,\chi_2) + \text{h.c.}\end{aligned}$$

- VEV configuration

$$\langle\phi_{1,2}\rangle = \frac{v_{1,2}}{\sqrt{2}}\ ,\quad \langle\phi_3\rangle = \frac{v_3}{\sqrt{2}}\ ,\quad (\langle\chi_1\rangle,\langle\chi_2\rangle)^t = \frac{W}{\sqrt{2}}\,(1,1)^t$$

- Preserved subgroups

$D_4 \xrightarrow{\langle\phi_1\rangle,\langle\phi_{2,3}\rangle} D_2$ in the charged lepton sector

$D_4 \xrightarrow{\langle\phi_1\rangle} D_4$ in the Dirac ν sector

$D_4 \xrightarrow{\langle\chi_i\rangle} Z_2$ in the Majorana ν sector

$\mu\tau$ Symmetry from Non-Trivial D_4 Breaking

- D_4 is not broken by $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$, since ϕ_1 and ϕ_2 are total singlets $\underline{\mathbf{1}}_{\mathbf{++}}$ of D_4
- D_4 is broken to D_2 by $\langle \phi_3 \rangle$, since ϕ_3 transforms as $\underline{\mathbf{1}}_{\mathbf{+-}}$ under D_4. It leaves invariant a group generated by $(hg)^2$ and g.
- D_4 is broken to Z_2, if $\langle \chi_i \rangle$ fulfills

$$\langle \chi_i \rangle \propto v_{+1} \quad \text{with} \quad h\, v_{+1} = +1\, v_{+1}$$

$$\text{i.e.} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Additionally, also a field transforming as $\underline{\mathbf{1}}_{\mathbf{-+}}$ preserves this Z_2 group.

$\mu\tau$ Symmetry from Non-Trivial D_4 Breaking

- Charged lepton mass matrix

$$m_e = |y_3\, v_1|/\sqrt{2}\,, \quad m_\mu = |y_4\, v_2 + y_5\, v_3|/\sqrt{2}\,, \quad m_\tau = |y_4\, v_2 - y_5\, v_3|/\sqrt{2}$$

- Dirac ν mass matrix

$$\mathcal{M}_\nu = \mathrm{diag}(a, b, b)\,, \quad a = y_1\, v_1^\star/\sqrt{2}\,, \quad b = y_2\, v_1^\star/\sqrt{2}$$

- Majorana mass for ν_Rs

$$M_{RR} = \begin{pmatrix} M & M_\chi & M_\chi \\ M_\chi & M' & 0 \\ M_\chi & 0 & M' \end{pmatrix} \quad \text{with} \quad M_\chi = \frac{y_\chi\, W}{\sqrt{2}}$$

- Additional results: Prediction of NH

  Relation $|m_{ee}| = \frac{m_1 m_2}{m_3}$

Subgroups of D_4

D_4	$D_2 = \langle (hg)^2, h \rangle$	$D_2 = \langle (hg)^2, g \rangle$	Z_4	$Z_2 = \langle h(hg)^m \rangle$ $m = 0, 2$	$Z_2 = \langle h(hg)^m \rangle$ $m = 1, 3$	Z_2
$\underline{\mathbf{1}}_{++}$	1	1	1	1	1	1
$\underline{\mathbf{1}}_{+-}$		1			1	1
$\underline{\mathbf{1}}_{-+}$	1			1		1
$\underline{\mathbf{1}}_{--}$			1			1
$\underline{\mathbf{2}}$				1 $+\ldots$	1 $+\ldots$	

with 1 being a total singlet of the subgroup

Variants of the Model

(e.g. Adulpravitchai et al. ('08))

- Model can be supersymmeterized
- Higgs doublets are replaced by flavons (vacuum is aligned)
- Maximal number of flavons allowed by subgroup acquires VEV
- No right-handed neutrinos present
- $Z_2^{(aux)}$ is replaced by Z_5
- Results in certain SCPV scenario
  - Only IH allowed

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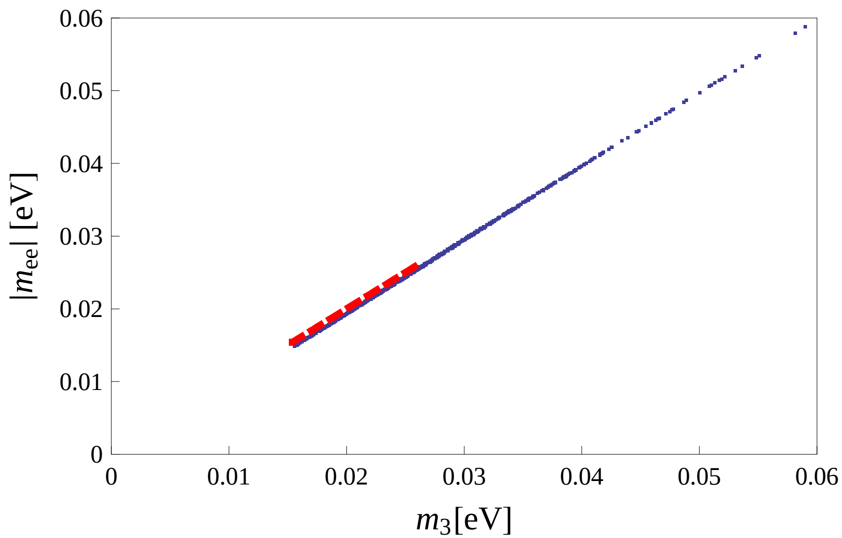
  - $|m_{ee}| \approx m_3$



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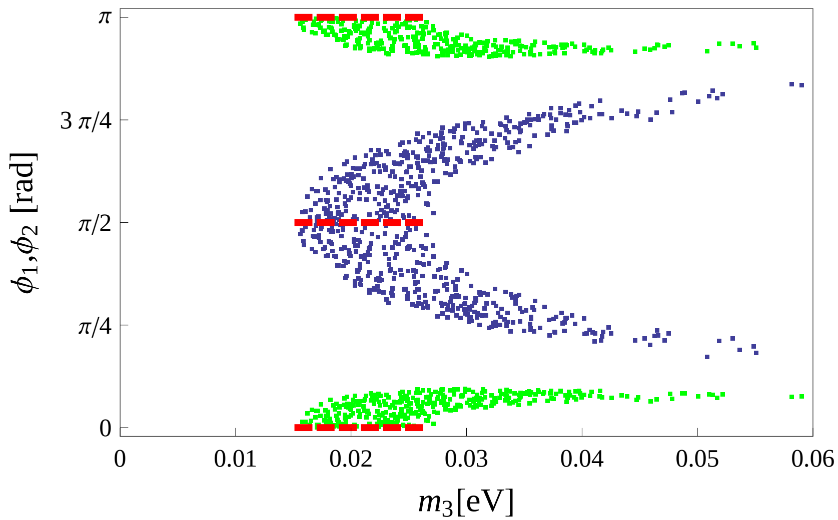

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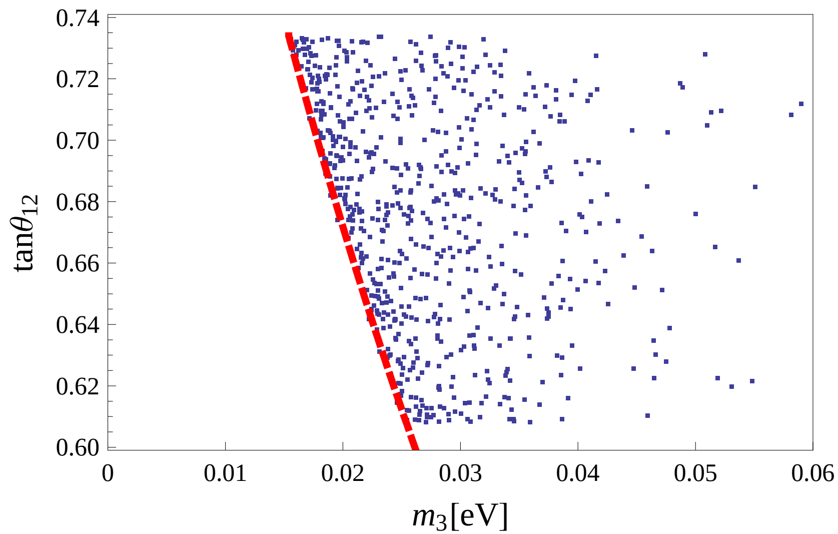
  - θ_{12} within 2σ range



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  - $|m_{ee}| \approx m_3$
  - Majorana phases $\phi_{1,2}$ are constrained
  - θ_{12} within 2σ range

Conclusions & Outlook

- Models with flavor symmetries, continuous or discrete, are successful in predicting TBM or $\mu\tau$ symmetry
- A_4 is the prime candidate for TBM, whereas small dihedral groups, e.g. D_4, are capable of predicting $\mu\tau$ symmetry
- Non-trivial breaking of flavor symmetry is crucial
- Phenomenological aspects such as leptogenesis and lepton flavor violating processes have been studied

Open questions

- Extension to the quark sector
- Embedding of the models into a GUT, preferably $SO(10)$
- Study of discrete anomalies in the existing models
- Study of the origin of the discrete symmetry

Thank you.

Back up

$\mu\tau$ Symmetry from Non-Trivial $D_3 \cong S_3$ Breaking

$\mu\tau$ Symmetry from Non-Trivial D_3 Breaking *(Grimus/Lavoura ('05))*

- Flavor symmetry is $D_3 \cong S_3$ with additional symmetry $Z_2^{(aux)}$
- Leptons

$$D_e \sim (\underline{\mathbf{1}}_{\mathbf{1}}, +) \; , \quad (D_\mu, D_\tau)^t \sim (\underline{\mathbf{2}}, +) \; ,$$
$$e_R \sim (\underline{\mathbf{1}}_{\mathbf{1}}, -) \; , \quad (\mu_R, \tau_R)^t \sim (\underline{\mathbf{2}}, +) \; ,$$
$$\nu_{e\,R} \sim (\underline{\mathbf{1}}_{\mathbf{1}}, -) \; , \quad (\nu_{\mu\,R}, \nu_{\tau\,R})^t \sim (\underline{\mathbf{2}}, -)$$

- Scalars (ϕ_i are $SU(2)_L$ doublets, χ gauge singlet)

$$\phi_1 \sim (\underline{\mathbf{1}}_{\mathbf{1}}, -) \; ,$$
$$\phi_2 \sim (\underline{\mathbf{1}}_{\mathbf{1}}, +) \; , \quad \phi_3 \sim (\underline{\mathbf{1}}_{\mathbf{2}}, +) \; ,$$
$$(\chi, \chi^\star)^t \sim (\underline{\mathbf{2}}, +)$$

Group Theory of D_3

- D_3 is the smallest discrete non-abelian group, belongs to the dihedral groups and is isomorphic to the permutation group S_3
- Its order is 6, i.e. it has six distinct elements
- It has 3 irred. reps., $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{1}}_{\mathbf{2}}$, $\underline{\mathbf{2}}$
- Kronecker products

$$\underline{\mathbf{1}}_{\mathbf{1}} \times \mu = \mu \quad \forall \quad \mu \,,$$
$$\underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{1}}_{\mathbf{2}} = \underline{\mathbf{1}}_{\mathbf{1}} \,, \quad \underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{2}} = \underline{\mathbf{2}} \,,$$
$$\underline{\mathbf{2}} \times \underline{\mathbf{2}} = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}$$

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- It has 3 irred. reps., $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{1}}_{\mathbf{2}}$, $\underline{\mathbf{2}}$
- Generator relations of D_3

$$\mathrm{A}^3 = \mathbb{1} \; , \;\; \mathrm{B}^2 = \mathbb{1} \; , \;\; \mathrm{A}\,\mathrm{B}\,\mathrm{A} = \mathrm{B}$$

- Generators

$$\mathrm{A} = +1 \quad , \quad \mathrm{B} = +1 \qquad \text{... for } \underline{\mathbf{1}}_{\mathbf{1}}$$

$$\mathrm{A} = +1 \quad , \quad \mathrm{B} = -1 \qquad \text{... for } \underline{\mathbf{1}}_{\mathbf{2}}$$

$$\mathrm{A} = \begin{pmatrix} \mathrm{e}^{\frac{2\pi i}{3}} & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi i}{3}} \end{pmatrix} \quad , \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{... for } \underline{\mathbf{2}}$$

$\mu\tau$ Symmetry from Non-Trivial D_3 Breaking

- Yukawa couplings

$$
\begin{aligned}
\mathcal{L}_Y &= y_1\, \overline{D}_e\, \nu_{e\,R}\, \phi_1^c + y_2\, (\overline{D}_\mu\, \nu_{\mu\,R} + \overline{D}_\tau\, \nu_{\tau\,R})\, \phi_1^c \\
&\quad + y_3\, \overline{D}_e\, e_R\, \phi_1 + y_4\, (\overline{D}_\mu\, \mu_R + \overline{D}_\tau\, \tau_R)\, \phi_2 + y_5\, (\overline{D}_\mu\, \mu_R - \overline{D}_\tau\, \tau_R)\, \phi_3 + \text{h.c.} \\
\mathcal{L}_{\nu_R} &= M\, \nu_{e\,R}\, \nu_{e\,R} + M'\, (\nu_{\mu\,R}\, \nu_{\tau\,R} + \nu_{\tau\,R}\, \nu_{\mu\,R}) \\
&\quad + y_\chi\, (\nu_{e\,R}\, \nu_{\mu\,R}\, \chi^\star + \nu_{e\,R}\, \nu_{\tau\,R}\, \chi) + y_\chi\, (\nu_{\mu\,R}\, \nu_{e\,R}\, \chi^\star + \nu_{\tau\,R}\, \nu_{e\,R}\, \chi) \\
&\quad + z_\chi\, (\nu_{\mu\,R}\, \nu_{\mu\,R}\, \chi + \nu_{\tau\,R}\, \nu_{\tau\,R}\, \chi^\star) + \text{h.c.}
\end{aligned}
$$

- VEV configuration

$$
\langle \phi_{1,2} \rangle = v_{1,2}\ , \quad \langle \phi_3 \rangle = v_3\ ,
$$

$$
\begin{pmatrix} \langle \chi \rangle \\ \langle \chi^\star \rangle \end{pmatrix} = \begin{pmatrix} W \\ W^\star \end{pmatrix} = |W| \begin{pmatrix} e^{i\,\alpha} \\ e^{-i\,\alpha} \end{pmatrix}
$$

$\mu\tau$ Symmetry from Non-Trivial D_3 Breaking

- Preserved subgroups

$$D_3 \xrightarrow{\langle\phi_1\rangle,\langle\phi_{2,3}\rangle} Z_3 \quad \text{in the charged lepton sector}$$

$$D_3 \xrightarrow{\langle\phi_1\rangle} D_3 \quad \text{in the Dirac } \nu \text{ sector}$$

$$D_3 \xrightarrow{\langle\chi\rangle,\langle\chi^\star\rangle} Z_2 \quad \text{in the Majorana } \nu \text{ sector}$$

for $\alpha = 0, \pm\frac{\pi}{3}, \pm\frac{2\,\pi}{3}, \pi$

i.e. $$\begin{pmatrix} \langle\chi\rangle \\ \langle\chi^\star\rangle \end{pmatrix} = |W|\,\mathrm{e}^{-i\,\alpha} \begin{pmatrix} \mathrm{e}^{2\,i\,\alpha} \\ 1 \end{pmatrix} \propto \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\alpha=0,\pm\pi}, \underbrace{\begin{pmatrix} \mathrm{e}^{-\frac{4\,\pi\,i}{3}} \\ 1 \end{pmatrix}}_{\alpha=\frac{\pi}{3},-\frac{2\,\pi}{3}}, \underbrace{\begin{pmatrix} \mathrm{e}^{-\frac{2\,\pi\,i}{3}} \\ 1 \end{pmatrix}}_{\alpha=-\frac{\pi}{3},\frac{2\,\pi}{3}}.$$

$\mu\tau$ Symmetry from Non-Trivial D_3 Breaking

- D_3 is not broken by $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$, since ϕ_1 and ϕ_2 are total singlets $\underline{\mathbf{1}}_{\mathbf{1}}$ of D_3
- D_3 is broken to Z_3 by $\langle \phi_3 \rangle$, since ϕ_3 transforms as $\underline{\mathbf{1}}_{\mathbf{2}}$ under D_3. It leaves invariant a group generated by A only.
- D_3 is broken to Z_2, if $\langle \chi \rangle$ and $\langle \chi^\star \rangle$ fulfill either

$$\begin{pmatrix} \langle \chi \rangle \\ \langle \chi^\star \rangle \end{pmatrix} \propto v_{+1} \quad \text{with} \quad \mathrm{B}\, v_{+1} = +1\, v_{+1}$$

$$\text{i.e.} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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$$\left(\begin{array}{c}\langle\chi\rangle\\ \langle\chi^\star\rangle\end{array}\right) \propto v_{+1} \quad \text{with} \quad \mathrm{B}\,\mathrm{A}\,v_{+1} = +1\,v_{+1}$$

i.e. $$\left(\begin{array}{cc} 0 & \mathrm{e}^{-\frac{2\,\pi\,i}{3}} \\ \mathrm{e}^{\frac{2\,\pi\,i}{3}} & 0 \end{array}\right) \left(\begin{array}{c} \mathrm{e}^{-\frac{2\,\pi\,i}{3}} \\ 1 \end{array}\right) = \left(\begin{array}{c} \mathrm{e}^{-\frac{2\,\pi\,i}{3}} \\ 1 \end{array}\right)$$

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$$\begin{pmatrix} \langle \chi \rangle \\ \langle \chi^\star \rangle \end{pmatrix} \propto v_{+1} \quad \text{with} \quad \mathrm{B}\,\mathrm{A}^2\, v_{+1} = +1\, v_{+1}$$

$$\text{i.e.} \quad \begin{pmatrix} 0 & \mathrm{e}^{-\frac{4\,\pi\,i}{3}} \\ \mathrm{e}^{\frac{4\,\pi\,i}{3}} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac{4\,\pi\,i}{3}} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\frac{4\,\pi\,i}{3}} \\ 1 \end{pmatrix}$$

$\mu\tau$ Symmetry from Non-Trivial D_3 Breaking

- Charged lepton mass matrix

$$m_e = |y_3\, v_1|\ ,\quad m_\mu = |y_4\, v_2 + y_5\, v_3|\ ,\quad m_\tau = |y_4\, v_2 - y_5\, v_3|$$

- Dirac ν mass matrix

$$\mathcal{M}_\nu = \mathrm{diag}(a, b, b)\ ,\quad a = y_1\, v_1^\star\ ,\quad b = y_2\, v_1^\star$$

- Majorana mass for ν_Rs (after rephasing)

$$M_{RR} = \begin{pmatrix} M & y_\chi\,|W| & y_\chi\,|W| \\ y_\chi\,|W| & z_\chi\,|W|\,\mathrm{e}^{3\,i\,\alpha} & M' \\ y_\chi\,|W| & M' & z_\chi\,|W|\,\mathrm{e}^{-3\,i\,\alpha} \end{pmatrix}$$

$\Rightarrow$ Light ν mass matrix M_ν is $\mu\tau$ symmetric, if $\alpha = 0, \pm\frac{\pi}{3}, \pm\frac{2\,\pi}{3}, \pi$

Subgroups of D_3

D_3	Z_3	$Z_2 = \langle \mathrm{BA}^m \rangle$ $m = 0, 1, 2$
$\underline{\mathbf{1}}_1$	$\mathbf{1}$	$\mathbf{1}$
$\underline{\mathbf{1}}_2$	$\mathbf{1}$	
$\underline{\mathbf{2}}$		$\mathbf{1} + ...$

with $\mathbf{1}$ being a total singlet of the subgroup