Black Hole Thermodynamics and Hawking Radiation

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The notation implied throughout is $c = G = k = 1$ (k is the Boltzmann constant) and except where specifically required for a result, $\hbar = 1$. Metric signature is $(-1, 1, 1, 1)$.

Black Hole Properties

Black hole are extremely dense objects which typically exhibit an event horizon. The definition is complicated a bit due to the possible notion of 'naked singularities', singularities not shrouded by an event horizon but for all intents and purposes (particularly this discussion) a black hole is a point mass surrounded by a space-time surface from which light cannot escape to spacial infinity. An event horizon does not automatically imply a singularity, only that a mass is within it's Chandrasekhar limit. For stellar objects, this is effectively the point of no return since no known mechanism of condensed matter can support such dense objects from total collapse. For truely huge matter distributions, an event horizon can be acheived without total collapse. Parking the entire Milky Way into a volume of radius 0.1 light years gives an average density approximately of water but it is within it's Chandrasekhar limit. There is a notion in cosmology that the visible universe has so much mass within it that the radius of 15 billion light years is less than the Chandrasekhar limit of the mass.

Metrics which describe 'nice' black holes are found by finding solutions to the Einstein Field Equations which have the surface of no escape for light (null geodesics which do not extend to spacial infinity).

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \tag{1}$$

$$G_{ab} = T_{ab} \tag{2}$$

The simplest and best known black hole solution is that of the Schwarzchild solution. It's the only black hole solution describing a spherically symmetric static space-time with zero cosmological constant Λ.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{3}$$

This can be generalised in three ways (assuming $\Lambda = 0$ and you are only doing four dimensional space-time). The first is to consider a charged mass which leads to the Reissner-Nordstrom solution, found in 1926. Another is the Kerr solution, which describes a spinning black hole. The third is a combination of the two, a spinning charged black hole, the Kerr-Newman metric, found in the mid 60s.

$$\begin{aligned} ds^2 &= (r^2 + a^2 \cos^2\theta) \left(\frac{dr^2}{r^2 - 2Mr + a^2 + Q^2} + d\theta^2 \right) \\ &+ \frac{1}{r^2 + a^2 \cos^2\theta} \Big(\sin^2\theta (a\, dt^2 + [r^2 + a^2] d\phi^2)^2 - (r^2 - 2Mr + a^2 + Q^2)(dt - a \sin^2\theta d\phi^2)^2 \Big) \end{aligned} \tag{4}$$

These three parameters, mass M, charge Q and rotation parameter a completely define the black hole. If two different black holes have the same values of these parameters they are indistinguishable at this level of description. Unfortunately, the understand the various black hole mechanics results derived by Hawking, the full Kerr-Newman metric (4) is required.

Zeroth Law of Black Hole Mechanics

Angular Velocity

The horizon, $\mathcal{H}$, is defined as the solution to $r^2 - 2Mr + a^2 + Q^2 = 0$, in the same manner as $r = 2M$ in a Schwarzchild metric, being the surface where the metric coefficent g_{rr} is singular.

The K-N metric (4) has two Killing vectors which give rise to symmetries of the system :

$$\begin{array}{lll} \text{Time translation} & t^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} & t^a = (1,0,0,0) \\ \text{Rotational translation} & r^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial \phi} & r^a = (0,0,0,1) \end{array}$$

A Killing vector k represents a symmetry of a system described by a Lagrangian L via the Lie derivative $\mathcal{L}_k L = 0$. For function f, $\mathcal{L}_k f = k^a \partial_a f$ and so the time translation Killing vector amounts to $\partial_t L = 0$, the energy of a black hole system is conserved and the rotational translation gives $\partial_\phi L = 0$, angular momentum is conversed. The Lie derivative $\mathcal{L}_X$ is linear in X and so we are able to consider the linear combination $k^a = t^a + \Omega r^a$.

$$k^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$$

This is null on the horizon for constant Ω, $k_a k^a|_{\mathcal{H}} = 0$. Therefore Ω defines the angular velocity of the black hole. The horizons are on $r_\pm$ with the outer horizon at r_+

$$\begin{aligned} r_+ &= M + \sqrt{M^2 - a^2 - Q^2} && (5) \\ \Rightarrow \quad \Omega &= \frac{a}{2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}} && (6) \end{aligned}$$

The angular velocity is constant on the horizon and independant of θ.

Surface gravity of a black hole

Since k^a is null on the horizon, it must obey the geodesic equations there. Therefore on $\mathcal{H}$ we have the following conditions on k

$$\begin{aligned} \nabla_a k_b + \nabla_b k_a &= 0 \\ k_a k^a &= 0 \\ k^a \nabla_a k^b &= \pm \kappa k^b \qquad \kappa \geq 0 && (7) \end{aligned}$$

Equation (7) can be reduced to $\kappa = 0$ if $k^a = \frac{dx^a}{du}$ where u is an affine parameter. For non-null trajectories this is possible but since k^a is null on this horizon, it is not possible, therefore $\kappa \neq 0$, has units of acceleration and is defined as the surface gravity. To calculate κ consider the defining properties of k^a away from the horizon :

$$\begin{aligned} k^a \nabla_a k^b &= \pm \kappa k^b \\ \Rightarrow \quad -k^a \nabla^b k_a &= \pm \kappa k^b \\ \Rightarrow \quad -\frac{1}{2} \nabla^b (k_a k^a) &= \pm \kappa k^b \qquad k_a k^a = -V && (8) \end{aligned}$$

To compute κ in the simplest way, a particular choice of coordinates is needed. In (t, r, θ, ϕ) coordinates $k^a = (1, 0, 0, 0)$. Then consider a change of coordinates to $v' = t + r + 2M \ln|r - 2M|$ so that $k^{a'} = A^{a'}_a k^a$ where $A^{a'}_a = \frac{\partial x^{a'}}{\partial x^a}$. Therefore

$$k^{v'} = A^{v'}_t k^t = 1 \quad \Rightarrow \quad k^{a'} = (1, 0, 0, 0) \qquad k_{a'} = g_{a'b'} k^{b'} = \left(-1 + \frac{2M}{r}, 1, 0, 0 \right)$$

The metric $g_{a'b'}$ is the Schwarzchild metric in the new (t, v, θ, ϕ) coordinate system :

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
\Rightarrow \quad k^{a'} k_{a'} &= -1 + \frac{2M}{r} \\
\Rightarrow \quad -\frac{1}{2}\frac{\partial}{\partial r}\left(-1 + \frac{2M}{r}\right) &= \pm\kappa \\
\frac{M}{r^2} &= \pm\kappa
\end{aligned}$$

On the Schwarzchild horizon $r = 2M$ this gives $\kappa = \frac{1}{4M}$. For the Kerr-Newman metric

$$\kappa = \frac{\sqrt{M^2 - a^a - Q^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}} \tag{9}$$

κ does not depend on θ, so it does not matter where on the horizon you are. This is known as the zero'th law of black hole mechanics. In the case of $M^2 = a^2 + Q^2$ $\kappa = 0$, this would equate to an 'extremal' black hole, which has lost it's event horizon.

First Law of Black Hole Mechanics

With the expression for the surface gravity derived, it is now of interest to express other physical quantities in terms of the three independent variables M, Q and a and the surface gravity.

Electrostatic potential of a black hole

For a charged black hole, $Q \neq 0$, which means that the standard electromagnetic vector potential A_a will also be non-zero. Define the Killing vector of time translations k^a by $k^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t}$. Since $A_0 = \Phi$, the electrostatic potential, this choice of Killing bector gives

$$\begin{aligned}
\Phi &= k^a A_a = \frac{Q}{r} \\
\Rightarrow \quad \Phi\big|_{\mathcal{H}} &= \frac{Q(M + \sqrt{M^2 - a^2 - Q^2})}{2M^2 - a^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}}
\end{aligned} \tag{10}$$

Angular momentum of a black hole

The angular momentum is defined simply as

$$J = aM \tag{11}$$

This makes clear the particular choice of a as a rotational parameter, not angular momentum or angular velocity.

Short Aside

It should be noted that a large amount of computation has been swept under the rug in defining the quantities J and Φ, indeed even M and Q. In the case of point-like objects, they are simple to define but should a distribution of charged matter been considered, the physical computation of even M is non-trivial and was not put on rigorous grounding until the 60s by Komar. The problems arise from the lack of a well defined notion of frequency or momentum when measuring a physical system in relativity, it is dependent on your choice of frame. As such, the quantities involved are not generally well defined unless you compute values at spacial infinity. The relevent integrals, having defined $dS^{ab} = t^{[a}r^{b]}d\Sigma$, are

$$
\begin{aligned}
M_K &= \frac{1}{4\pi}\int_{S^2|_\infty} \nabla_a k_b dS^{ab} = \frac{1}{4\pi}\int_\Sigma (2T_{bc} - Tg_{bc})k^b d\Sigma^c \qquad k^a\frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} \\
Q &= \frac{1}{4\pi}\int_{S^2|_\iota} F_{ab}dS^{ab} \\
J_K &= \frac{1}{8\pi}\int_{S^2|_\infty} \nabla_a m_b dS^{ab} = aM \qquad m^a\frac{\partial}{\partial x^a} = \frac{\partial}{\partial \phi}
\end{aligned}
$$

The required physical quantities are defined in terms of measurable quantities like F_{ab}. In each case, for the simple distribution of a black hole, these reduce to the M, Q and $J = aM$ in the Kerr-Newman metric.

Event Horizon Area

The event horizon of a black hole is dependent upon the mass, charge and rotation of the black hole. For a 2-sphere of constant r and t the line element length has an induced metric of the form

$$dl^2 = g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 = h_{ab}dx^a dx^b \tag{12}$$

The area of the sphere is then found by the standard geometric result

$$
\begin{aligned}
A(r) &= \int_{S^2} \sqrt{h}\, d\Sigma \\
&= \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{(r^2+a^2)^2 - a^2(r^2 - 2Mr + a^2)}\sin\theta \\
&= 4\pi\sqrt{(r^2+a^2)^2 - a^2(r^2 - 2Mr + a^2)} \qquad (13)\\
A\big|_{\mathcal{H}} &= 4\pi(r_+^2 + a^2) = 4\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}) \qquad (14)
\end{aligned}
$$

Consider changing a black hole's state by altering the variables from (M, Q, J) to $(M+\delta M, Q+\delta Q, J+\delta J)$. The area of the event horizon $A = A(M, J, Q)$ therefore undergoes an infinitesimal change. The chain rule gives

$$dA = \left(\frac{\partial A}{\partial M}\right)dM + \left(\frac{\partial A}{\partial Q}\right)dQ + \left(\frac{\partial A}{\partial J}\right)dJ \tag{15}$$

For the Kerr-Newman metric

$$
\begin{aligned}
\left(\frac{\partial A}{\partial M}\right) = \frac{8\pi}{\kappa} \qquad \left(\frac{\partial A}{\partial J}\right) &= -\frac{8\pi\Omega}{\kappa} \qquad \left(\frac{\partial A}{\partial Q}\right) = -\frac{8\pi\Phi}{\kappa} \\
dA &= \frac{8\pi}{\kappa}dM - \frac{8\pi\Omega}{\kappa}dJ - \frac{8\pi\Phi}{\kappa}dQ \\
dM &= \frac{\kappa}{8\pi}dA + \Omega dJ + \Phi dQ \qquad (16)
\end{aligned}
$$

This gives an equivalence between $\frac{\kappa}{8\pi}dA$ and TdS from thermodynamics by comparing with the First Law of Thermodynamics

$$dE = T\,dS - P\,dV \tag{17}$$

Of course, just because two equations look similar doesn't automatically infer a deep connection. A little bit more is needed before putting too much faith in what might be nothing more than a coincidence. However, with this hint, a particular avenue of investigation is opened up. In thermodynamics $dS \geq 0$ in a closed system. Can the same be said for a black hole?

The Second Law of Black Hole Mechanics

In thermodynamics, S changes if E does, so consider adding matter to a black hole and it's effect on the event horizon area. The simplest way to add matter or it's relativistic equivalent, energy, is to fire a photon into the black hole. This gives us the convenience of considering the electromagnetic field tensor. Start from Maxwell's equations to describe light propogation

$$\nabla_a F^{ab} = 0 \qquad F_{ab} = \nabla_a A_b - \nabla_b A_a$$

Recall there exists a gauge transformation of A_a, $A_a \rightarrow A_{a'} = A_a + \nabla_a \Lambda$. These leave F_{ab} invariant and give a wave equation for A_a

$$\nabla_a F^{ab} = \nabla_a(\nabla^a A^b - \nabla^b A^a) = \Box A^b - \nabla_a \nabla^b A^a$$

Given our gauge freedom it is useful to pick the gauge $\nabla_a A^a = 0$, reducing this to an expression involving the space-time curvature (ie the Ricci tensor),

$$\begin{aligned} \Box A^b - R_a{}^{bac} A_c - \underbrace{\nabla_b \nabla_a A^a}_{=0} &= 0 \\ \Rightarrow \quad \Box A^b - R^{bc} A_c &= 0 \end{aligned} \tag{18}$$

In this derivation, it is assumed that the wavelength of light is much less than the curvature scale, otherwise problems involving quantum gravity come into effect. Planck scale effects have been shown to be unimportant in this derivation but the workings are much more complex.

In trying to find a solution to this equation, consider the approximate solution in flat space-time as $A_a = a_a e^{\frac{i\theta}{\hbar}}$ with $\theta = p_a x^a$ which defines the 4-momentum of the light ray. a_a is the polarisation vector and in flat space-time it does not depend on position.

The momentum operator is $-i\hbar\partial_a = \hat{p}_a$, so $\hat{p}_c A_a = p_c A_a$. Two conditions exist on the wave function, the wave equation implies $p^2 = 0$ and the gauge condition implies $a_a p^a = 0$. This method can be adapted for curved space-time in a manner very similar to the common 'series solution' method of solving ODEs.

$$A_a = (a_a + \hbar b_a + \hbar^2 c_a + ...)e^{\frac{i\theta}{\hbar}} \tag{19}$$

The gauge choice now becomes

$$\nabla_a A^a \quad = \quad (\nabla_a a^a + \hbar \nabla_a b^a + \hbar^2 \nabla_a c^a + ...)e^{\frac{i\theta}{\hbar}} + \frac{1}{\hbar}(a^a + \hbar b^a + \hbar^2 c^a + ...)\nabla_a \theta \, e^{\frac{i\theta}{\hbar}} = 0 \tag{20}$$

Considering term by term equations in $\hbar$

$$\begin{array}{ll} O(\hbar^{-1}) & a^a \nabla_a \theta = 0 \\ O(\hbar) & \nabla_a a^a + i p_a b^a = 0 \end{array}$$

The Maxwell equation $\Box A^a - R^{ab} A_b = 0$ expands to

$$\begin{aligned} (\Box a^a + \hbar \Box b^a + ...) + \frac{i}{\hbar}\nabla_b \nabla^b \theta (a^a + \hbar b^a + ...) - \frac{1}{\hbar^2}(\nabla\theta)^2(a^a + \hbar b^a + ...) \\ + \quad \frac{2}{\hbar}\nabla^b \theta (\nabla_b a^a + \hbar \nabla_b b^a + ...) - R^{ab}(a_b + \hbar b_b + ...) \quad &= \quad 0 \end{aligned} \tag{21}$$

Considering term by term equations in $\hbar$

$$\begin{array}{ll} O(\hbar^{-2}) & (\nabla_a \theta)^2 = p^2 = 0 \\ O(\hbar^{-1}) & a_a \nabla_b p^b + 2 p^b \nabla_b a_a + p_b p^b b_a = 0 \end{array}$$

Since $p_a p^a = 0$, the trajectory is null and using $\nabla_b p_a = \nabla_a p_b$ (since $p_a = \partial_a \theta$) the geodesic equation $p^a \nabla_a p_b = 0$ is derived. This also shows that a_a contains information about amplitude and polarisation,

$$p^b \nabla_b a_a = -\frac{1}{2} a_a \nabla_b p^b$$

Express a_a as $a t_a$ where a is the amplitude and t_a the unit space-like polarisation vector ($t^a \nabla_b t_a = 0$).

$$\begin{aligned} p^b \nabla_b (a t_a) &= -\frac{1}{2} a t_a \nabla_b p^b \\ p^b (\nabla_b a) t_a + p^b a \nabla_b t_a &= -\frac{1}{2} a t_a \nabla_b p^b \end{aligned}$$

Contract with t^a

$$\begin{array}{rll} & p^b \nabla_b a = -\frac{1}{2} a \nabla_b p^b & \text{Amplitude Equation} \\ \Rightarrow & p^b \nabla_b t_a = 0 & \text{Polarisation Equation} \end{array} \quad (22)$$

These two equations give a description of the electromagnetic field's development through curved space-time. Despite the $O(\hbar^n)$ terms involving R^{ab} not being considered, the derivative is covariant, so the metric connection is in all equations and the curvature of space-time

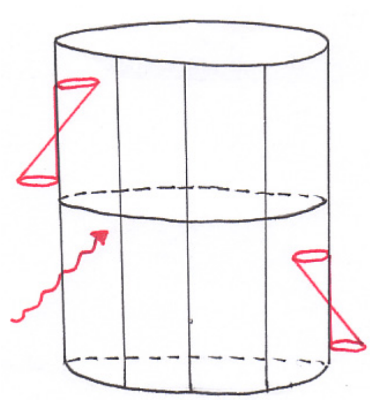
Consider a beam of photons moving through curved space-time. The beam length is parameterised by λ and has cross sectional area $A(\lambda)$. The energy of the beam will be proportional to the area and the photon strength, $\propto A^a A_a$. Therefore $E \propto Aa^2$ and will be conserved in terms varying λ where $\partial_\lambda = p^a \nabla_a$. This gives the following :

$$\begin{array}{rrcl} & \frac{d}{d\lambda}(Aa^2) & = & 0 \\ \Rightarrow & p^a \nabla_a (Aa^2) & = & 0 \\ & & = & p^a (\nabla_a A) a^2 + 2 p^a (\nabla_a a) A \\ & & = & p^a (\nabla_a A) a^2 - a^2 \nabla_a p^a A \end{array} \quad (23)$$

Therefore the area of the beam obeys the equation

$$p^a \nabla_a A = A \nabla_a p^a \quad (24)$$

Consider the event horizon of a black hole. The picture below has some dimensions supressed, but the event horizon's can be seen as being related to the cross sectional area of the cylinder. The equations derived now apply to how this cross section's swept out volume is 'focused' by the adding of matter.



Two quantities need to be considered, first and second derivatives of the event horizon area in time, which is again parameterised by λ. The second derivative equation is known as 'The Focussing Equation'. If $dA \geq 0$, the Focusing Equation needs to be such that $A'' \leq 0$ ($' = \partial_\lambda$). For convenience $\sqrt{A}$ is considered instead.

$$\begin{aligned}
\frac{d^2}{d\lambda^2}\sqrt{A} &= p^a\nabla_a(p^b\nabla_b\sqrt{A}) && (25)\\
&= p^a\nabla_a\left(\frac{1}{2}\frac{1}{\sqrt{A}}p^b\nabla_b A\right)\\
&= \frac{1}{2}p^a\nabla_a\left(\sqrt{A}\nabla_b p^b\right)\\
&= \frac{1}{4}p^a(\nabla_a A)\frac{1}{\sqrt{A}}\nabla_b p^b + \frac{1}{2}\sqrt{A}p^a\nabla_a\nabla_b p^b\\
&= \frac{1}{4}\sqrt{A}\nabla_a p^a\nabla_b p^b + \frac{1}{2}\sqrt{A}p^a\nabla_b\nabla_a p^b + \frac{1}{2}\sqrt{A}p^a R_{ab}{}^b{}_c p^c\\
&= \frac{1}{4}\sqrt{A}\nabla_a p^a\nabla_b p^b - \frac{1}{2}\sqrt{A}R_{ab}p^a p^b + \frac{1}{2}\sqrt{A}p^a\nabla_b\nabla_a p^b\\
&= \frac{1}{4}\sqrt{A}\nabla_a p^a\nabla_b p^b - \frac{1}{2}\sqrt{A}R_{ab}p^a p^b + \sqrt{1}2\sqrt{A}\nabla_b\underbrace{(p^a\nabla_a p^b)}_{=0} - \frac{1}{2}\sqrt{A}(\nabla_b p_a)(\nabla^a p^b)\\
&= -\sqrt{A}\left(\frac{1}{2}R_{ab}p^a p^b + \sigma^2\right) && (26)\\
\text{where}\quad \sigma^2 &= \frac{1}{2}\nabla_a p_a\nabla^b p^a - \frac{1}{4}\nabla_a p^a\nabla_b p^b\\
&= \frac{1}{2}\nabla_a p_b\nabla_c p_d\left(g^{ac}g^{bd} - \frac{1}{2}g^{ab}g^{cd}\right) && (27)
\end{aligned}$$

If we are wanting to find $A'' \leq 0$, then both terms in (26) are required to be positive. To check the positivity of σ^2, define the tensor X_{ab} by $X_{ab} = \nabla_a p_b$ and choose coordinates such that $p_b = (s, -s, 0, 0)$ which gives the general symmetric form of X_{ab} as the following :

$$X_{ab} = \begin{pmatrix} \alpha & \alpha & \beta & \gamma \\ \alpha & \alpha & \beta & \gamma \\ \beta & \beta & \epsilon & \lambda \\ \gamma & \gamma & \lambda & \xi \end{pmatrix}$$

Since we are checking the positivity of σ^2 at a specific space-time point we are able to pick the local coordinate frame $g_{ab} = \eta_{ab}$ which causes σ^2 to simplify down to

$$\sigma^2 = \lambda^2 + \frac{1}{2}(\epsilon - \xi)^2 \geq 0$$

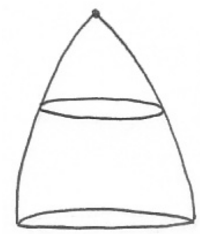
Therefore if the term in the Focusing Equation involving R_{ab} is positive, the required result is found. Using the Einstein Field Equations we can express the Ricci tensor in terms of the energy-momentum tensor, $R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$ and recall that $0 = p_a p^a = g_{ab}p^a p^b$, which gives :

$$\begin{aligned}
R_{ab}p^a p^b &= 8\pi T_{ab}p^a p^b && (28)\\
T_{ab} &= \begin{pmatrix} \text{Energy Density} & \text{Momentum Flux} \\ \text{"} & \begin{bmatrix} \text{Stress} \end{bmatrix} \end{pmatrix}\\
\Rightarrow\quad T_{ab}p^a p^b &= \text{Energy density} + \left|\text{Momentum flux}\right| > 0\\
\Rightarrow\quad R_{ab}p^a p^b &\geq 0 \qquad \text{Strong energy condition} && (29)
\end{aligned}$$

Thus the Focusing equation becomes the required result,

$$\frac{d^2}{d\lambda^2}\sqrt{A} \leq 0 \tag{30}$$

This equation is used in all the singularity theorems. However, this is not a guarentee that $dA \geq 0$ always, given certain behaviour of A'. Suppose now that $A \to 0$, while the Focusing Equation holds. This would lead to the following behaviour of the event horizon :



The event horizon has vanished, yet the singularity remains, known as a 'naked singularity'. Therefore it can be reasonably supposed that this does not happen. Consider $\frac{d}{d\lambda}\sqrt{A} < 0$ at the instant matter falls into the black hole. This implies $\sqrt{A} \to 0$ at some point in the future. Thus $\frac{d}{d\lambda}\sqrt{A} \geq 0$ or else naked singularities exist.

From this work the relation between A in black hole mechanics and S in thermodynamics have been further validated. Not only do they obey similar relations with the other variables of their respective systems, they always increase during physical processes.

Overview of the Three Laws

The three black hole laws are

0th Law	$\partial_\theta \kappa = 0$
1st Law	$dM = \frac{\kappa}{8\pi} dA + \Phi dQ + \Omega dJ$
2nd Law	$dA \geq 0$

This strongly suggests that $A \sim S$ and $\kappa \sim T$. Classically a black hole's temperature is zero, due to no thermal radiation, so entropy must be infinite. This is resolved by $T \sim \hbar\kappa$ and $S \sim \frac{A}{\hbar}$ so classically $T = 0$ and $S = \infty$ for a black hole, it's a perfect sink for energy.

While it is possible to extract some energy from a rotating black hole via the Penrose process (akin to space probes stealing velocity from planets' gravity wells), this only works for rotating black holes and at best can only remove a small percentage of the black hole's energy before all it's angular momentum is removed. As such, generally $dM \geq 0$, relativistically for non-rotating black holes the passage of energy and mass is one way, once inside the event horizon it will never leave and even rotating black holes can only shed approximately 17% of their mass. Relativity is just one side of the story though. While quantum mechanics does not usually play much of a role in systems which relativity describes, the compact yet massive nature of a black hole means both should be considered, especially when a mechanism to explain how exactly a black hole could have temperature is needed. This is because any object with non-zero temperature must radiate energy, but in relativity's eyes matter and energy go into a black hole, not out. Therefore this suggests a phenomena not entirely within the realms of relativity is occuring.

Relativity and Quantum Theory

Hawking Radiation and Classical Field Theory

The simplest way to see how curved space-time can alter the behaviour of a quantum field is to consider a scalar field in a static space-time. For a free scalar field ϕ, there is the Lagrangian

$$L = \int d^4x \sqrt{g} \left(-\frac{1}{2}\partial_a\phi\partial_b\phi g^{ab} - \frac{1}{2}m^2\phi^2 \right) \tag{31}$$

Variation with respect to ϕ gives the Klein-Gordon equation, though the d'Alembert operator is defined with the curved space-time metric.

$$(-\Box + m^2)\phi = 0$$

For simplicity, consider static space-times which have a metric of the form $ds^2 = -V^2(x)dt^2 + h_{ij}dx^i dx^j$. This leads to solutions looking like those of standard quantum field theory,

$$\phi_a \sim e^{-i\omega_n t}\frac{\chi_a(x)}{\sqrt{2|\omega_n|}} \tag{32}$$

The positive frequency (or energy) then follows, with the energy operator $i\hbar\partial_t$. The energy of this mode is $\hbar\omega_n$. Thus $\omega_n > 0$ is a positive frequency (energy) and $\omega_n < 0$ is negative. Substituting into the Klein-Gordon equation gives specific equations for the χ_n which does not reduce to the standard QFT equation for the energy mode coefficents due to the $V(x)$ metric entry.

$$\begin{aligned}
\frac{1}{V^2}\frac{\partial^2\phi}{\partial t^2} - \Box i\omega\phi - h^{ij}(\partial_i \ln V)\partial_j\phi - \frac{m^2}{\hbar^2} &= 0 \\
\Rightarrow \quad -\Box_{(n)}\chi_n - \frac{\omega_n^2\chi_n}{V^2} - h^{ij}(\partial_i \ln V)\partial_j\chi_n + \frac{m^2}{\hbar^2}\chi_n &= 0
\end{aligned} \tag{33}$$

The field can be expressed as a sum of the individual mode solutions.

$$\phi = \sum_n a_n\phi_n + a_n^\dagger\phi_n^*$$

This is still a classical viewpoint and so the theory needs to be quantised, though an inner product on the space of solutions is required. Note that there is a curved space-time equivalent of a Noether Current,

$$j^a = i(\phi^*\nabla^a\phi - \phi\nabla^a\phi^*)$$

$\nabla^a j_a = 0$ is a consequence of equations of motion, as expected of a Noether Current. Define a Hermitian inner product as

$$\langle f, g\rangle = \int_\Sigma (f^*\nabla_a g - g\nabla_a f^*)d\Sigma^a \qquad \langle \bar{f}, \bar{f}\rangle \;=\; -\langle f, f\rangle \qquad \langle \bar{f}, f\rangle = 0$$

The failure of positivity is what makes standard quantum mechanics not work. It needs to be replaced by the field theory. We need to divide all modes into positive frequency and negative frequency

$$\begin{aligned}
p_n &= \tfrac{e^{-i\omega_n t}}{\sqrt{2|\omega_n|}}\chi_n(x) \quad \omega_n > 0 \\
n_n &= \tfrac{e^{i\omega_n t}}{\sqrt{2|\omega_n|}}\chi_n(x) \quad \omega_n < 0
\end{aligned}$$

Orthogonality for spacial part of the modes follows from the form of the differential equation governing χ_n,

$$\int \sqrt{h}d^3x\frac{1}{V}\chi_n^*\chi_m = \delta_{nm} \tag{34}$$

So for the Klein-Gordon norm

$$\langle p_n, p_m\rangle = i\int_\Sigma \left(\frac{e^{i\omega_n t}}{\sqrt{2\omega_n}}\chi_n^*\nabla_a\left(\frac{e^{-i\omega_m t}}{\sqrt{2\omega_m}}\chi_m^*\right) - (n \leftrightarrow m)\right)d\Sigma^a = \delta_{nm}$$

We have made use of $d\Sigma^a \propto \sqrt{h}d^3x\frac{1}{V}$. Similarly $\langle n_m, n_n\rangle = \delta_{nm}$ and $\langle p_m, n_n\rangle = 0$. Now the system can be quantised, with these orthogonality results. The action is given as,

$$I = \int \sqrt{g}\underbrace{d^4x}_{\sqrt{h}d^3xdt}\Big(\underbrace{-\frac{1}{2}\partial_a\partial_b\phi g^{ab}}_{\frac{1}{2V^2}\dot{\phi}^2} - \frac{1}{2}\frac{m^2}{\hbar^2}\phi^2\Big) \tag{35}$$

Define the canonical momentum

$$\pi = \frac{\delta I}{\delta \dot{\phi}} = \frac{\sqrt{h}}{V}\dot{\phi}$$

Use the Heisenberg Picture and define

$$\left[\phi(\mathbf{x}), \pi(\mathbf{y})\right] = i\hbar\delta(\mathbf{x}^1 - \mathbf{y}^1)\delta(\mathbf{x}^2 - \mathbf{y}^2)\delta(\mathbf{x}^3 - \mathbf{y}^3)$$

Note that

$$\begin{aligned} f(\mathbf{y}) &= \int \sqrt{h}\frac{d^3x}{V}\delta^3(\mathbf{x}-\mathbf{y})f(x) \\ \Rightarrow \quad \delta^3(\mathbf{x}-\mathbf{y}) &= \frac{\delta(\mathbf{x}^1 - \mathbf{y}^1)\delta(\mathbf{x}^2 - \mathbf{y}^2)\delta(\mathbf{x}^3 - \mathbf{y}^3)}{\sqrt{h}}V(x) \end{aligned}$$

This is a tensor density of weight 1 on Σ, so

$$\left[\phi(\mathbf{x}), \dot{\phi}(\mathbf{y})\right] = i\hbar\delta^3(\mathbf{x}-\mathbf{y})$$

Expressing the field in terms of positive and negative modes

$$\phi = \sum_n a_n p_n + a_n^\dagger \qquad \dot{\phi} = \sum_n (-i\omega_n a_n p_n + i\omega_n a_n^\dagger n_n)$$

Substituting into the commutator yeilds

$$\sum_{n,m}\left[a_n p_n(\mathbf{x}) + a_n^\dagger(\mathbf{x}), -i\omega_m a_m p_m(\mathbf{y}) + i\omega_m a_m^\dagger n_m(\mathbf{y})\right] = i\hbar\delta^3(\mathbf{x}-\mathbf{y})$$

Take the Klein-Gordon inner product with $p_r(\mathbf{x})$ and $p_s(\mathbf{y})$ allows for specific relations to be picked out. This gives

$$\left[a_n, a_m\right] = 0 \qquad \left[a_n^\dagger, a_m^\dagger\right] = 0 \qquad \left[a_n, a_m^\dagger\right] = \delta_{nm} \tag{36}$$

This is like Minkowski space, so the interpretation is that a Fock space description is possible and that there exists a vacuum state $|0\rangle$ and n_i particles in the i'th mode can be given by

$$|n_1, n_2, ..., n_i\rangle = \Pi_j \frac{(a_j^\dagger)^{n_j}}{\sqrt{(n_j)!}}|0\rangle$$

Consider the field coefficents in the distant past, p_{-k} and n_{-k} and in the distant future, p_i and n_i. The distant past and future are choosen because the system is static then, but inbetween the system can be time dependent, such as black hole colliding or matter falling into a black hole, where the specific dynamics are too complex to solve. Since we are not interested in the dynamics, only their finished state we can avoid computing specifics about such time dependent behaviours. Consider we are only interested how the fields change overall. If nothing happens, the inner product of the fields will not change, orthogonality is conserved. If not, the set of inner products will evolve into more complex relations than just $\langle q_n, q_m\rangle = \delta_{mn}$ (q either p or n). Express such a new set of relations as follows,

$$\begin{aligned} \langle p_{-k}, p_i\rangle &= A_{ki} \\ \langle p_{-k}, n_i\rangle &= C_{ki} \\ \langle n_{-k}, p_i\rangle &= -B_{ki} \\ \langle n_{-k}, n_i\rangle &= -D_{ki} \end{aligned}$$

The overall field ϕ can be expressed in terms of both distant past and distant future modes, giving $\phi = \sum_j a_j p_{-j} + a_j^\dagger n_{-j} = \sum_j b_j p_j + b_j^\dagger n_j$. Using the inner product to pick out the annihilation and creation operators from the past gives,

$$\begin{aligned}\langle p_{-i}, \phi\rangle \quad \Rightarrow \quad a_i &= \sum_j b_j(p_{-i}, p_j) + b_j^\dagger(p_{-i}, n_j) \\ &= \sum_j b_j A_{ij} + b_j^\dagger C_{ij} \qquad (37)\\ \langle p_{-i}, \phi\rangle \quad \Rightarrow \quad a_i^\dagger &= \sum_j b_j(-B_{ij}) + b_j^\dagger(-D_{ij}) \qquad (38)\end{aligned}$$

Since a_i is the Hermitian conjugate of $a_i^\dagger$, we have

$$A^* = -D^T \qquad C^* = -B^T$$

Similarly, future operators are given as,

$$\langle p_i, \phi\rangle \quad \Rightarrow \quad b_i = \sum_j a_j A_{ij}^* + a_j^*(-B_{ij}^*) \qquad (39)$$

$$\langle n_i, \phi\rangle \quad \Rightarrow \quad b_i^\dagger = \sum_j C_{ij}^* + a_j^*(-D_{ij}^*) \qquad (40)$$

Substituting these equations into one another gives consistency conditions on A, B, C and D. We have

$$b_i = \sum_{j,k}(b_k A_{ik} + b_k^\dagger C_{ik})A_{ij}^* + (b_k B_{jk} + b_k^* D_{jk})B_{ij}^*$$

And so we must have

$$\begin{aligned}\sum_{j,k}(A_{jk}A_{ij}^* + B_{jk}B_{ij}^*)b_k &= b_i \\ \Rightarrow \sum_{j,k}(A_{jk}A_{ij}^* + B_{jk}B_{ij}^*)b_k &= \delta_{ik} \Leftrightarrow A^*A + B^*B = I \qquad (41)\end{aligned}$$

Similarly, the coefficent of b_k^* vanishes giving $A^*C + B^*D = 0$. These are basically expressions of unitarity, something which is to be expected (and indeed, required) for a physically meaningful system such as this.

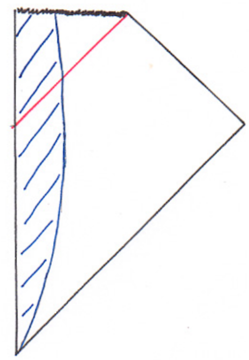
Now, using the operators on our Fock space vacuum (in the distance past), $|0_-\rangle$, the evolution matrices A, B, C and D allow for an expression of the expected number of particles in a given field mode in the future,

$$\begin{aligned}N_s &= \langle 0_-|b_s^\dagger b_s|0_-\rangle \\ &= \langle 0_-|\left(\sum_k a_k^\dagger A_{sk} - a_k B_{sk}\right)\left(\sum_j a_j A_{sj}^* - a_j^\dagger B_{sj}^*\right)|0_-\rangle \\ &= \sum_{j,k} B_{sk}B_{sj}^*\langle 0_-|a_k a_j^*|0_-\rangle \\ &= \sum_{j,k} B_{sk}B_{sj}^*\langle 0_-|\delta_{jk} + a_j^* a_k|0_-\rangle \\ &= \sum_j B_{sj}B_{sj}^* = \sum_j |B_{sj}|^2 \geq 0 \qquad (42)\end{aligned}$$

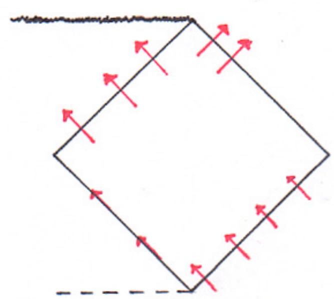
The basic point is that particle creation is caused by modes having no definition of true frequency for all time. A system which involves matter and complex interactions of gravitational fields is expected to produce particles of radiation.

Particle Emission and Absorption

Consider the gravitational collapse of a collection of matter to form a black hole, as represented by the Penrose diagram below,



Radiation comes from the distance past, some of which falls into the resultant black hole and some of which is emitted by the collapsing material and detected in the distance future. The Schwarzchild diagram with radiation emitted in the distant past and during the collapse and being measured in the distance future at null infinity gives the following diagram.



The vacuum on $\mathcal{H}^-$ is defined with respect to positive frequency relativity to the Kruskal time coordinate U. Observations at $\mathcal{J}^+$ require true frequency with respect $u = t - r - 2M\ln|r-2M|$,

$$U = -e^{-\kappa u}$$

where κ is the surface gravity of the black hole. For simplicity take $\kappa = \frac{1}{4M}$ for the Schwarzchild black hole. Therefore

$$\begin{aligned} u &= -\frac{1}{\kappa}\ln(-U) \\ \Rightarrow \quad e^{-i\omega u} &\sim e^{\frac{i\omega}{\kappa}\ln(-U)} \sim (-U)^{\frac{i\omega}{\kappa}} \qquad U<0 \end{aligned}$$

Using a Fourier series allows for a splitting of the positive and negative energy moves, as before.

$$(-U)^{\frac{i\omega}{\kappa}}\theta(-U) = \underbrace{\int_0^\infty \alpha_{\omega\omega'} e^{-i\omega' U} d\omega'}_{\text{positive frequency}} + \underbrace{\int_0^\infty \beta_{\omega\omega'} e^{i\omega' U} d\omega'}_{\text{negative frequency}} \tag{43}$$

Multiply by $e^{i\omega'' U}$ and integrate over U extracts specific mode coefficents due to orthogonality.

$$RHS \quad = \quad 2\pi\int_0^{2\pi}\Big(\delta(\omega'-\omega'')\alpha_{\omega\omega'} + \delta(\omega'+\omega'')\beta_{\omega\omega''}\Big)d\omega$$

$$
\begin{aligned}
&= 2\pi\alpha_{\omega\omega''} \quad \text{or} \quad 2\pi\beta_{\omega,-\omega''} && (44)\\
LHS &= \int_{-\infty}^{\infty} \theta(-U)(-U)^{\frac{i\omega}{\kappa}} e^{-\omega'' U} dU \\
&= \int_{\infty}^{\infty} (-U)^{\frac{i\omega}{\kappa}} e^{i\omega'' U} dU = \begin{cases} \alpha & \omega'' < 0 \\ \beta & \omega'' > 0 \end{cases} && (45)
\end{aligned}
$$

There is a branch point in the integral at $\omega'' = 0$. This gives the result

$$
\begin{aligned}
\frac{|\beta_{\omega,-\omega''}|}{|\alpha_{\omega,\omega''}|} &= e^{-\frac{\pi\omega}{\kappa}} \\
\beta &\sim \text{emission of particles} \\
\alpha &\sim \text{absorption of particles} \\
\frac{\text{Prob. of emission of a particle}}{\text{Prob. of absorption of a particle}} &= \left|\frac{\beta}{\alpha}\right|^2 = e^{-\frac{2\pi\omega}{\kappa}} && (46)
\end{aligned}
$$

Comparing with well established thermodynamic results, this is the expected result for a black body interacting with particles of energy E and it's temperature $T \sim e^{-\frac{E}{T}}$, same if $T = \frac{\kappa}{2\pi}$. For the case of the Schwarzchild solution $T = \frac{\kappa}{2\pi}$.

Black Hole Lifetime

If black holes emit energy, then if they are not being fed by infalling matter, they will eventually evaporate if such processes occur. Even when considering the cosmic microwave background, which bathes the universe in a 2.7K 'glow', this is a decreasing universal temperature and will eventually be colder than even the lowest temperature of black holes (though it might take trillions of trillions of years!). Though for simplicity, such an energy source is ignored and space can be assumed to be zero Kelvin.

To compute the expected lifetime of a black hole of mass M, recall the first law of blackhole mechanics

$$
\begin{aligned}
dM &= \frac{\kappa}{8\pi} dA + \Phi dQ + \Omega dJ \\
&\simeq T dS + \Phi dQ + \Omega dJ \\
\Rightarrow \quad T &= \frac{\hbar\kappa}{2\pi} \qquad S = \frac{A}{4\hbar}
\end{aligned}
$$

For a Schwarzchild black hole $\kappa = \frac{1}{4M}$ so $T = \frac{1}{8\pi M}$.

$$
\begin{aligned}
M = 1M_\odot = 10^{30}\text{kg} \qquad & T = 10^{-7} \\
10^{12}\text{kg} \qquad & T = 10^{12}
\end{aligned}
$$

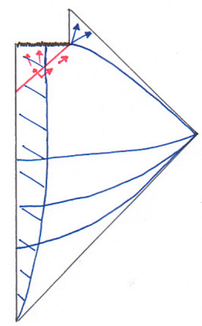
Infact, this is an instability of black holes, the more they radiate, the hotter they get, increasing their rate of energy emission.

$$
\text{Rate of radiation} = \sum_{m<T} \{\text{Radiation energy density}\} \times \{\text{Cross sectional area}\}
$$

A black hole will radiate a particle of mass m if $kT >> mc^2$. This is one of the signatures a black hole at the LHC might leave, an even spread of particle types.

$$
\begin{aligned}
-\dot{M} &= c\frac{1}{M^4} M^2 \qquad c \sim O(1) \\
\Rightarrow \quad -\dot{M}M^2 &= c \\
M(t) &= \sqrt{M_0^3 - 3ct} \qquad M(0) = M_0 && (47)
\end{aligned}
$$

So $M \to 0$ after a time $\frac{M_0^3}{3c}$, the black hole disappears. For a black hole the mass of the Sun, it's about 10^{54} years. For one smaller, though not formed by gravitational collapse but perhaps by pressures at the beginning of the universe a mass of $10^{12} kg$ would mean it would be evaporating about now, given a lifetime of approximately 10^{10} years. A Penrose diagram can be drawn for an evaporating black hole



A system develops in time according to the transformation $|i\rangle \to e^{iHt}|i\rangle$. If quantum mechanics works, there is a unitary transformation that takes $|i\rangle$ to $|f\rangle$. So no unitary transformation exists because of the boundary given by the singularity.

Three types of proposal to deal with this.

1. Modify quantum mechanics
2. Picture is wrong and black holes do not evaporate, they leave a remenant
3. Even though there is a boundary, no information gets lost.

The second law of black hole mechanics gives $\Delta A \geq 0$ provided $R_{ab}k^a k^b \geq 0 \; \forall k^a$ null. This gives $T_{ab}k^a k^b \geq 0$, therefore Energy density $\geq$ |Energy flux|. However, this does not necessarily hold for quantum fields. This is why we see $\Delta A \leq 0$ due to Hawking radiation, in this case $T_{ab}k^a k^b \leq 0$.

Euclideanisation and Canonical Ensembles

Using quantum field theory operators is not the only way to get Hawking's result. Canonical ensembles, common place in thermodynamics, also provide a method for such analysis. Consider the usual form of the Schwarzchild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Using only the zeroth law of thermodynamics, if thermal equilibrium is achieved at temperature T, then everything is at temperature T. Consider free scalar fields satisfying $(-\Box + m^2)\phi = 0$ with mode functions $e^{-i\omega t} Y_{l,m}(\theta, \phi) f_{n,l}(r)$. The function f satisfies a confluent form of the equation. Look for a Greens function

$$(-\Box + m^2)G = \delta$$

Because the signature is $(- + ++)$, then $\Box$ is a wave operator. In flat space-time the solution is choosen using the Euclidean Postulate, the Wick rotation,

$$t \to -i\tau \quad \text{gives a metric signature} \quad (+ + ++)$$

The solution of $(-\Box + m^2)G = \delta$ is now unique because $-\Box$ is elliptic. Now we calculate $G(x, x'; \tau - \tau')$ (only depends on t difference, since ∂_t is Killing vector). To extract a physical answer, send $\tau \to -it$.

The Euclidean-Schwarzchild metric is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$r = 2M$ looks singular. Note that $r > 2M$ has a signature $(+\ +\ +\ +)$ and $r < 2M$ has signature $(-\ -\ +\ +)$. Introduce a new radial coordinate ρ

$$r = 2M + \frac{\rho^2}{8M}$$

Consider coordinates close to $\rho = 0$, just above the event horizon,

$$\begin{aligned} 1 - \frac{2M}{r} &\approx 1 - 2M\left(2M + \frac{\rho^2}{8M}\right)^{-1} \sim \frac{\rho^2}{16M^2} \\ dr &= \frac{\rho d\rho}{4M} \qquad |\rho| << 1 \end{aligned}$$

Therefore the metric becomes

$$\begin{aligned} ds^2 &= \frac{\rho^2}{16M^2} d\tau^2 + \frac{\rho^2 d\rho^2 16M^2}{16M^2 \rho^2} + 4M^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= d\rho^2 + \rho^2 d\left(\frac{\tau}{4M}\right)^2 + \underbrace{(2M)^2(d\theta^2 + \sin^2\theta d\phi^2)}_{S^2|_{R=2M}} \end{aligned} \tag{48}$$

It follows that $\frac{\tau}{4M}$ has to behave like an angle, since the second term in the metric is of the form $r^2 d\theta^2$, as seen in 2d polar coordinates. If $\frac{\tau}{4M}$ is not identified with $\frac{\tau}{4M} + 2\pi$ then have conical singularity.

Considering vacuum solutions $R_{ab} = 0$, no canonical singularity is implied. $\frac{\tau}{4M}$ is thus a polar angle and should be identified with period 2π, $\tau \to \tau + 8\pi M$. Therefore the Euclidean Schwarzchild metric gives

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$\begin{array}{lll} \infty > r > 2M & \text{Riemannian} & \\ 0 < \tau < 8\pi M & \text{Metric } \mathbb{R}^2 \times S^2 & R_{ab} = 0 \end{array}$$

The Green's function G is then well defined with the periodicity condition of

$$G(x, x'; \tau - \tau') = G(x, x', \tau - \tau' + 8\pi M) \qquad n \in \mathbb{Z}$$

Consider now the expectation values of an observable O and the quantum field ϕ in terms of the inverse temperature β and Hamiltonian H :

$$\begin{aligned} \langle O \rangle &= \frac{\mathrm{Tr}(e^{\beta H} O)}{\mathrm{Tr}(e^{\beta H})} \\ G(\mathbf{x}, \mathbf{x}', t - t') = \langle \phi(\mathbf{x}, t)\phi(\mathbf{x}', t') \rangle &= \frac{\mathrm{Tr}\big(e^{\beta H}\phi(\mathbf{x}, t)\phi(\mathbf{x}', t')\big)}{\mathrm{Tr}(e^{\beta H})} \end{aligned} \tag{49}$$

Work in the Heisenberg Picture, so that states do not depend on time, only operators do. The time translation operator is therefore

$$\phi(\mathbf{x}, t + t_0) = e^{-Ht_0}\phi(\mathbf{x}, t) e^{iHt_0}$$

Put $t_0 = i\beta$, so that

$$\phi(\mathbf{x}, t + i\beta) = e^{\beta H}\phi(\mathbf{x}, t) e^{-\beta H}$$

And so,

$$\begin{aligned}
G(\mathbf{x},\mathbf{x}',t-t'+i\beta) &= \mathrm{Tr}\big(e^{-\beta H}\phi(\mathbf{x},t+i\beta)\phi(\mathbf{x}',t')\big)/... \\
&= \mathrm{Tr}\big(e^{-\beta H}e^{\beta H}\phi(\mathbf{x},t)e^{-\beta H}\phi(\mathbf{x}',t')\big)/... \\
&= \mathrm{Tr}\big(e^{-\beta H}\phi(\mathbf{x}',t')\phi(\mathbf{x},t)\big)/... \\
&= \mathrm{Tr}\big(e^{-\beta H}\phi(\mathbf{x},t)\phi(\mathbf{x}',t')\big)/... \qquad \big[\phi,\phi'\big]=0 \\
&= G(\mathbf{x},\mathbf{x}';t-t') \qquad (50)
\end{aligned}$$

The Greens function in a canonical ensemble has periodicity $i\beta$ (sometimes called the KMS condition).

Combining this with the periodicity found in τ for the Schwarzchild metric and the zeroth law of thermodynamics gives the temperature of the black hole is $T = \frac{1}{8\pi M}$.