QCD factorization in $B$ decays – ten years later

M. Beneke (RWTH Aachen)

Flavianet meeting in honour of Chris Sachrajda on the occasion of his 60th birthday
Southampton, England, December 14/15, 2009

Outline

• Early years of QCD factorization
  MB, G. Buchalla, M. Neubert and C.T. Sachrajda,

• The next order: Tree-dominated decays at NNLO
  MB, T. Huber, X.Q. Li, 0911.3655 [hep-ph];
15 years of common scientific interests

15 years of common scientific interests


- Exclusive B decays and QCD factorization (1998–2009–???)

MB, G. Buchalla, M. Neubert and C.T. Sachrajda,

*QCD factorization for B → \( \pi \pi \) decays: Strong phases and CP violation in the heavy quark limit*, Phys.Rev.Lett.83:1914-1917,1999 [hep-ph/9905312]


QCD Factorization for $B \to \pi\pi$ Decays: Strong Phases and CP Violation in the Heavy Quark Limit


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2Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309
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(Received 17 May 1999)
QCD Factorization for $B \to \pi\pi$ Decays:
In the Heavy Quark Limit


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2 Stanford Linear Accelerator Center, Stanford University
3 Department of Physics and Astronomy, University of Southern California

(Received 17 May 2009)

The most popular assumption is naive factorization

\[ \langle \pi\pi \rangle \sim (\bar{u}b)_{V-A} (\bar{d}u)_{V-A} \bar{B}_d \]

\[ = - i \int \frac{f_{B \to \pi}(m_{\pi}^2)}{m_{\pi}^2} \cdot M_B^2 \]

What about "non-factorizable" terms?
The most popular assumption is naive factorization

\[ \langle \pi \pi \rangle \cdot (\bar{u}b)_{V-A} \cdot (\bar{d}u)_{V-A} \cdot B_d \]

\[ = i f_\pi \frac{B \to \pi}{f(m_{\pi}^2)} \cdot M_B^2 \]

What about "non-factorizable" terms?

Ciuchini, Franco, Machelli, Silvestrini

"The state of the art in the calculation of the matrix elements is such that this turns out to be impossible."
Dear Matthias and Martin,

I hope that you both had a good end to your US trips and a safe trip home. I enjoyed our discussions in Minneapolis very much and as always learned a lot from them. As promised, this note contains some simple comments on factorization, further to our discussion in the Days Inn last Sunday. I sketch in a simple example how I believe that factorization might be established for class-1 decays, and you will see that, particularly in the treatment of the collinear mass singularities, I disagree with the comments you were making.

It will be clear below that nothing is proved to all orders. Up to now I have been simply trying to understand how Dugan-Grinstein might work in the Feynman gauge. Before talking to you last week, I felt I had a picture for how it worked for class-1 decays (summarised below) but couldn't see how it could possibly be true for class-2 decays (which I had planned to study further). Let me also add the obvious comments that these are notes written for you; I hope that they are clear but I have not made any effort to be complete or polished.

Chris, 25 Apr 1998
Dear Matthias and Martin,

I hope that you both had a good end to your US trips and a safe trip home. I enjoyed our discussions in Minneapolis very much and as always learned a lot from them. As promised, this note contains some simple comments on factorization, further to our discussion last Sunday. I sketch in a simple example how I think factorization might be established for class-I decays. I see that, particularly in the treatment of the collinear singularities, I disagree with the comments you wrote.

It will be clear below that nothing is proved to be true. I have been simply trying to understand how Dugan-Gribov factorization might work for class-I decays (sum rules). I couldn't see how it could possibly be true for class-II decays. If you like, I had planned to study further. Let me also add that these are notes written for you; I hope that you have not made any effort to be complete or polished.

Chris, 25 Apr 1998

MB, 26 Apr 1998

dear Chris,

this is only to acknowledge that I received your text, as I have read it only once so far.

my first impression is that all your arguments are perfectly valid and I do not remember having made a statement to the contrary in Minneapolis. However, there is one diagram for which collinear singularities do not cancel, I believe, and this is the exchange of a gluon between the two light quarks + wave function renormalization. This does not spoil factorization in the sense of a decoupling of the ud from the bc. This singularity just builds up the pion wave function. The correct factorization formula for class I decays seems to me the result of amending the pion wave function in the final state factor.

I am eager to work on this and Gerhard (Buchalla) is, too. As for me I have to finish a few things that I had to lay to the side while writing my thesis. They are all less interesting than exclusive B decays, but they have to be done, in particular as students are involved. I hope they can be done fast or at least leave some extra time.

With best regards, Martin
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MB, 29 Apr 1998

We would certainly be happy if you joined us on this project. My rough idea of it is this:

(a) Understand what the factorization formula is (this seems almost clear) and find some all-order arguments in favour of it.
(b) Do a complete NLO calculation of the hard scattering amplitudes.
(c) Understand at what order the soft/Feynman mechanism enters.
(d) Understand what happens for class 1 decays with two light mesons and class 2 decays.

Best regards, Martin
its light-cone distribution amplitude. At leading power in $\Lambda_{\text{QCD}}/m_b$, we find that the soft interactions can be summarized by the factorization formula

$$\langle \pi(p')\pi(q)|Q_i|\bar{B}(p)\rangle = f^{B\rightarrow \pi}(q^2) \int_0^1 dx\ T_i^1(x)\Phi_\pi(x) + \int_0^1 d\xi\ dx\ dy\ T_i^{11}(\xi,x,y)\Phi_\pi(\xi)\Phi_\pi(x)\Phi_\pi(y), \quad (2)$$
its light-cone distribution amplitude. At leading power in $\Lambda_{QCD}/m_b$, we find that the soft interactions can be described by light-cone distribution amplitudes. This discussion can be summarized by the factorization formula

$$\langle \pi(p')\pi(q)|Q_2|\tilde{B}(p)\rangle = f^{B\to\pi}(q^2) \int_0^1 dx T^i_1(x)\Phi_\pi(x) + \int_0^1 d\xi \int_0^1 dy T^i_{11}(\xi,x,y)\Phi_B(\xi)\Phi_\pi(x)\Phi_\pi(y), \quad (2)$$

Main result: in the heavy quark limit all "non-factorizable" diagrams are dominated by hard gluons / quarks and can be calculated as expansion in $1/d_s(m_b)$. Soft gluons are suppressed as $\Lambda_{QCD}/m_b$. QCD factorization.
its light-cone distribution amplitude. At leading power in $\Lambda_{QCD}/m_b$, we find that the soft interactions can be summarized by the factorization formula

$$\langle \pi(p')\pi(q)|Q_i|\bar{B}(p)\rangle = f^{B \rightarrow \pi}(q^2) \int_0^1 dx \, T^\pi(x) \Phi_\pi(x) + \int_0^1 d\xi \, dx \, dy \, T^{\pi\pi}(\xi,x,y) \Phi_B(\xi) \Phi_\pi(x) \Phi_\pi(y),$$

(2)

The approach discussed here allows us to formulate, for the first time, rigorous predictions of QCD for exclusive nonleptonic $B$ decays in the heavy quark limit. On the other hand, as the dependence on the formally power-suppressed coefficient $a_0^B(\pi \pi)$ demonstrates, the asymptotic limit may be problematic at $m_b \approx 5$ GeV and the applicability of the theory has to be decided on a case-by-case basis. The most important power corrections are those that depend on the chirally enhanced combination $t_X$. The $\alpha_s$ corrections to all such terms can in fact be identified. However, the factorization formula breaks down in this case, because the relevant twist-3 wave functions do not fall off fast enough at the end points. A detailed discussion of this will be given elsewhere.
Now consider the region in which $k$ is soft rather than supersoft, and again we illustrate the power counting in diagram 1a of Fig. 15. In this case the phase space is of $O(\lambda^8)$ and if the outer gluon is the soft one then the propagators scale as:

$$\frac{1}{\lambda^2} \frac{1}{\lambda^3} \frac{1}{\lambda} \frac{1}{\lambda^0} \frac{1}{\lambda} \frac{1}{\lambda},$$

which gives a combined factor of $O(1/\lambda^7)$. We therefore have no divergence from this region of phase-space (nor from the region in which the inner gluon is the soft one). This is not the case for all the diagrams, however, as we shall demonstrate below.

We now consider the supersoft–collinear and soft–collinear regions in turn.

5.5.2. The supersoft–collinear region

**Diagrams 1a–7b.** We start by considering the 18 diagrams 1a–7b. We place them into 4 groups \{1a, 3a; 2a, 6a; 4a, 7b\}, \{1b, 3b; 2b, 6b; 4b, 7b\}, \{1c, 6c; 2c, 5c; 3a, 7a\} and \{1d, 6d; 2d, 5d; 3b, 7a\}. We label these groups I–IV, and only consider explicitly the

**Diag 8a**

$$\text{Diag 8a} = \frac{4\alpha q \cdot (k + l) + 2(u + \alpha)q \cdot k + 2(uq + l)^2}{(uq + k + l)^2 k^2 l^2 (k + l)^2} \frac{p_b \cdot q}{(p_b \cdot k)(q \cdot k)},$$

**Diag 12a**

$$\text{Diag 12a} = \frac{2l^2}{(uq + l)^2 (uq + k + l)^2 k^2 l^2} \frac{p_b \cdot q}{(p_b \cdot k)(q \cdot k)},$$

**Diag 13a**

$$\text{Diag 13a} = \frac{-2q \cdot (i + k)}{(uq + l + k)^2 k^2 l^2} \frac{p_b \cdot q}{(p_b \cdot k)(q \cdot k)^2},$$

**Completed early September 1999**
Many questions and further theoretical developments

- All-order proofs, especially for light-light
- Is hard-scattering enough? Role of Sudakov suppression


Abstract

In order to obtain fundamental information about the Standard Model of particle physics from experimental measurements of exclusive hadronic two-body $B$-decays we have to be able to quantify the non-perturbative QCD effects. Although approaches based on the factorization of mass singularities into hadronic distribution amplitudes and form factors provide a rigorous theoretical framework for the evaluation of these effects in the heavy quark limit, it is not possible to calculate the $O(\Lambda_{QCD}/m_b)$ corrections in a model-independent way, because of the presence of non-factorizing long-distance contributions. It has been argued that Sudakov effects suppress these contributions and render the corresponding corrections perturbatively calculable in terms of the distribution amplitudes. In this paper we examine this claim for the simple and related example of semileptonic $B \rightarrow \tau$ decays (which have similar long-distance contributions) and conclude that it is not justified. The uncertainties in our knowledge of the mesons' distribution amplitudes imply that the calculations of the form factors are not sufficiently precise to be useful phenomenologically. Moreover, it appears that a significant fraction of the contribution comes from the non-perturbative region of large impact parameters, and is therefore incalculable. We also raise a number of theoretical issues in the derivation of the underlying formalism. Our conclusion is therefore a disappointing one. For $B$-decays it is not possible to invoke Sudakov effects to calculate amplitudes for decays which have long-distance divergences (end-point singularities) in the standard hard-scattering approach.
Many questions and further theoretical developments

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- Hard-spectator-scattering: role of the intermediate scale $\sqrt{m_b \Lambda}$? What is the $B$ meson LCDA? Does one need transverse-momentum dependence?
  
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- Hard-spectator-scattering: role of the intermediate scale $\sqrt{m_b\Lambda}$? What is the $B$ meson LCDA? Does one need transverse-momentum dependence?
  

Eventually the relevance of collinear modes (rather than soft only as in HQET) played an important role in the development of soft-collinear effective theory.
First data – failures and successes

Talks in 1999 concluded with statements like “Let’s hope that the heavy quark limit is good enough for the real world with $m_b = 5 \text{ GeV}$ and $\Lambda = 500 \text{ MeV}$.”
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Problems

- 2001: First measured direct CP asymmetries ($\pi^+K^-$, then $\pi^+\pi^-$ in 2002) opposite in sign to perturbative prediction. Annihilation? Other power corrections? Or NNLO?

$$A_{CP} = [c \times \alpha_s]_{NLO} + O(\alpha_s^2, \Lambda/m_b)$$
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\[
A_{CP} = [c \times \alpha_s]_{NLO} + \mathcal{O}(\alpha_s^2, \Lambda/m_b)
\]

- 2003: Colour-suppressed mode \( \pi^0 \pi^0 \) larger than prediction

Input parameters (\( \lambda_B \))? Power corrections? Or NNLO?

\[
C \propto a_2(\pi\pi) = 0.220 - [0.179 + 0.077i]_{NLO} + \left[ \frac{r_{sp}}{0.485} \right] \{[0.123]_{LOsp} + [0.072]_{tw3}\}
\]

\[
r_{sp} = \frac{9f_{M1}^2}{m_b f_{B\pi}^2(0) \lambda_B}
\]
Successes

- Colour-allowed and penguin-dominated branching fractions are usually quantitatively ok
  \[ PP \text{ vs } PV \text{ vs } VP \text{ (with some reservations), } \eta^{(r)}K^{(*)} \]
- CP asymmetries would have no reason to be small, if the amplitudes were completely
  long-distance dominated
- \[\gamma \approx (70 \pm 5)^\circ\] from time-dependent CP asymmetries
  Small \(|V_{ub}|\)
- Qualitative explanation of the absence of longitudinal polarization in \(B \to VV\) decays
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  Small $|V_{ub}|$

- Qualitative explanation of the absence of longitudinal polarization in $B \rightarrow VV$ decays
Why NNLO?

Basic question:

\[ \mathcal{O}(\alpha_s) \quad \text{vs} \quad \mathcal{O}\left(\frac{\Lambda}{m_b}\right) \]

Recall: problems with direct \( A_{\text{CP}} \) and some colour-suppressed modes are based on comparison to (essentially) leading order QCD predictions. Can never exclude large sizeable power corrections theoretically.
Basic question:

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Recall: problems with direct \( A_{CP} \) and some colour-suppressed modes are based on comparison to (essentially) leading order QCD predictions.

Can never exclude large sizeable power corrections theoretically.

NNLO or fit!
Understanding hard spectator-scattering

Fields & Scales

Electroweak + QCD + X

\[ \text{integrate out } W, z, t, X \quad k \approx M_W \]

QCD + QED

\[ k^2 \sim \Lambda^2 \]

SCET_1

\[ \text{integrate out hard scale } k \sim \Lambda \]

(\( \Lambda \) could be \( m_b \))

\[ k \sim (\Lambda, \sqrt{\Lambda}, \Lambda) \]

\[ n \cdot k + \frac{p_1 \cdot k}{2} + \frac{n \cdot k^2}{2} \]

SCET_II

describes \( k^2 \sim \Lambda^2 \)

collinear: \( k \sim (\Lambda, \Lambda, \Lambda) \)

soft: \( k \sim (\Lambda, \Lambda, \Lambda) \)

Fields

<table>
<thead>
<tr>
<th>light quark</th>
<th>gluon</th>
<th>heavy quark</th>
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<tbody>
<tr>
<td>( \xi_c )</td>
<td>( A_{bc} )</td>
<td>-</td>
</tr>
<tr>
<td>hard-collinear</td>
<td></td>
<td></td>
</tr>
<tr>
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</tr>
<tr>
<td>soft</td>
<td>( q_s )</td>
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</table>

If energetic particles move in different directions \( n_i \): need collinear fields for every direction.
Integrating out the scale $m_b$ [ QCD $\Rightarrow$ SCET$_{\text{I}}$]

Which hard subgraphs

$\left(\bar{b}b\right)\left(\bar{u}u\right) \rightarrow [\bar{X}^{(0)}_\left(\text{collinear} \left(m_b\right)\right) + \left(C_\text{I}^{\bar{u}b} \left[ \bar{b}g(s_n)h_n \right] + C_\text{II}^{\bar{u}b} \left[ \bar{b}s_{

Result:

- $M_2$ factorizes at scales $\mu < m_b$
- Strong phases $\bar{u}i$ perturbative coefficient functions $C_{\text{I,II}}$ ONLY
- Leaves out $1/m_b$ corrections (see below)
SCET_I → SCET_{II} \quad (\text{integrate out scale } \sqrt{m_\Lambda})

\begin{align*}
[\bar{q}(s_n,h_v)] & \quad \text{is an example, where naive SCET_{II} factorization is wrong} \\
\text{- keep this in SCET_I} & \quad (\text{or use } F^{\text{BH}(0)} \sim <M_1\bar{u}_1b_1\bar{B}> \text{ in QCD})
\end{align*}

\begin{align*}
[\bar{q}(s_n,h_v)A_{1hc}(s_2n_1)h_v] & \quad \rightarrow \quad J * \left[ \bar{q}(s_n,h_v) \right]_{\text{soft}} \left[ \bar{q}(s_n,h_v) \theta_0 \right]_{\text{collinear}} \\
\uparrow & \quad \phi_B \\
\downarrow & \quad \phi_{M_1}
\end{align*}

\text{J contains the hard-collinear spectator interactions}

Power counting implies that convolution integrals converge, and that only four-quark operators appear \( \sim \) product of 2-particle eight-cone distribution amplitudes

Checked by 1-loop calculations
( HB, Kiyo, Yang; Hill et al; HB, Yang)
Both together

\[
A(B \to H_1 H_2) = \text{factor} \times \sum_{\text{terms}} C(M_h) \times \left\{ \frac{f_{BM_1}}{(0)} \cdot T^{I \Pi}_{(M_H, M_S)} \ast f_{H_1} \phi_{H_1}(M_S) + \sqrt{m_b} \right. \\
\left. \frac{m_b}{\sqrt{m_b}} \ast \left[ T^{I \Pi}_{(M_H, M_1)} \ast J(M_1, M_S) \right] + f_{H_1} \phi_{H_1}(M_S) \ast f_{H_2} \phi_{H_2}(M_S) + \right. \\
\left. \frac{1}{m_b} \text{power corrections} \right\}
\]

- Precise prescriptions for performing higher-order calculations
- Large logs in C, J, T can be summed by RGEs

→ phenomenology
The next order


Two-loop correction to form factor term:
One-loop correction to $T_{ii}^{\Pi}(u, z)$:
One-loop correction to $J(z; \omega, \nu)$
NNLO Status

- **Spectator-scattering**
  - One-loop $J^{II}$: (Becher et al. ’04; MB, Yang ’05; Kirilin ’05)
  - One-loop $T^{II}$ tree amplitudes: (MB, Jäger, ’05; Kivel, ’06; Pilipp ’07)
  - One-loop $T^{II}$ penguin amplitudes: (MB, Jäger, ’06)

- **Vertex term**
  - Two-loop $T^{II}$ tree amplitudes: (MB, Huber, Li ’09; Bell ’09)
  - Two-loop $T^{II}$ penguin amplitudes: (in progress)

Tree-dominated decays complete at NNLO.
No NNLO result yet on direct CP asymmetries.
Size of the 2-loop vertex correction
Numerical result (tree amplitudes)

\[ a_1(\pi\pi) = 1.009 + [0.023 + 0.010i]_{\text{NLO}} + [0.026 + 0.028i]_{\text{NNLO}} - \left[ \frac{r_{\text{sp}}}{0.445} \right] \left\{ [0.014]_{\text{LOsp}} + [0.034 + 0.027i]_{\text{NLOsp}} + [0.008]_{\text{tw3}} \right\} \\
= 1.00 + 0.01i \quad \rightarrow \quad 0.93 - 0.02i \quad (\text{if } 2 \times r_{\text{sp}}) \]

\[ a_2(\pi\pi) = 0.220 - [0.179 + 0.077i]_{\text{NLO}} - [0.031 + 0.050i]_{\text{NNLO}} + \left[ \frac{r_{\text{sp}}}{0.445} \right] \left\{ [0.114]_{\text{LOsp}} + [0.049 + 0.051i]_{\text{NLOsp}} + [0.067]_{\text{tw3}} \right\} \\
= 0.24 - 0.08i \quad \rightarrow \quad 0.51 - 0.02i \quad (\text{if } 2 \times r_{\text{sp}}) \]

- The NNLO corrections to the vertex term and spectator scattering are significant individually (about 25% for \(a_2\)). But both tend to cancel – too bad!
- Largest uncertainty is input parameter dependence: Allows \(|C/T|_{\pi\pi} \approx 0.7\), if \(\lambda_B\) is small. The colour-suppressed amplitudes are probably dominated by spectator-scattering. But \(\text{arg}(C/T_{\pi\pi}) \lesssim 15^\circ\).
- Perturbation theory works at scale \(m_b\) and \(\sqrt{m_b\Lambda}\). No indication of further large radiative corrections.
### Branching fraction results (tree-dominated decays)

<table>
<thead>
<tr>
<th></th>
<th>Theory I</th>
<th>Theory II</th>
<th>Experiment</th>
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<tbody>
<tr>
<td>$B^- \to \pi^- \pi^0$</td>
<td>$5.43 \pm 0.06 \pm 1.45$</td>
<td>$5.82 \pm 0.07 \pm 1.42$</td>
<td>$5.59 \pm 0.41$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \to \pi^+ \pi^-$</td>
<td>$7.37 \pm 0.86 \pm 1.22$</td>
<td>$5.70 \pm 0.70 \pm 1.16$</td>
<td>$5.16 \pm 0.22$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \to \pi^0 \pi^0$</td>
<td>$0.33 \pm 0.11 \pm 0.42$</td>
<td>$0.63 \pm 0.12 \pm 0.64$</td>
<td>$1.55 \pm 0.19$</td>
</tr>
<tr>
<td>$B^- \to \pi^- \rho^0$</td>
<td>$8.68 \pm 0.42 \pm 2.71$</td>
<td>$9.84 \pm 0.41 \pm 2.54$</td>
<td>$8.3 \pm 1.2$</td>
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<tr>
<td>$B^- \to \pi^0 \rho^-$</td>
<td>$12.38 \pm 0.90 \pm 2.18$</td>
<td>$12.13 \pm 0.85 \pm 2.23$</td>
<td>$10.9 \pm 1.4$</td>
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<td>$\bar{B}_d^0 \to \pi^+ \rho^-$</td>
<td>$17.80 \pm 0.62 \pm 1.76$</td>
<td>$13.76 \pm 0.49 \pm 1.77$</td>
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<tr>
<td>$\bar{B}_d^0 \to \pi^- \rho^+$</td>
<td>$10.28 \pm 0.39 \pm 1.37$</td>
<td>$8.14 \pm 0.34 \pm 1.35$</td>
<td>$7.3 \pm 1.2$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \to \pi^\pm \rho^\mp$</td>
<td>$28.08 \pm 0.27 \pm 3.82$</td>
<td>$21.90 \pm 0.20 \pm 3.06$</td>
<td>$23.0 \pm 2.3$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \to \pi^0 \rho^0$</td>
<td>$0.52 \pm 0.04 \pm 1.11$</td>
<td>$1.49 \pm 0.07 \pm 1.77$</td>
<td>$2.0 \pm 0.5$</td>
</tr>
<tr>
<td>$B^- \to \rho_L^- \rho_L^0$</td>
<td>$18.42 \pm 0.23 \pm 3.92$</td>
<td>$19.06 \pm 0.24 \pm 4.59$</td>
<td>$22.8 \pm 1.8$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \to \rho_L^+ \rho_L^-$</td>
<td>$25.98 \pm 0.85 \pm 2.93$</td>
<td>$20.66 \pm 0.68 \pm 2.99$</td>
<td>$23.7 \pm 3.1$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \to \rho_L^0 \rho_L^0$</td>
<td>$0.39 \pm 0.03 \pm 0.83$</td>
<td>$1.05 \pm 0.05 \pm 1.62$</td>
<td>$0.55 \pm 0.22$</td>
</tr>
</tbody>
</table>
Factorization test

\[ \frac{\Gamma(B^- \to \pi^- \pi^0)}{d\Gamma(B^0 \to \pi^+ l^- \bar{\nu})/dq^2}\bigg|_{q^2=0} = \frac{3\pi^2 f_\pi^2 |V_{ud}|^2 |\alpha_1(\pi\pi) + \alpha_2(\pi\pi)|^2}{\Lambda_B^3} \]

- From semi-leptonic data
  [cf. Becher, Hill, '05; Ball '06, BaBar '06]:
  \[ |V_{ub}| f_+(0) = (9.1 \pm 0.7) \times 10^{-4} \]
  equivalent to
  \[ |\alpha_1(\pi\pi) + \alpha_2(\pi\pi)|_{\text{exp}} = 1.29 \pm 0.11 \]

- Leading uncertainties: \( \lambda_B \) (\( B \) LCDA), \( \alpha_2^\pi \) (pion LCDA), power corrections, hard-collinear scale-dependence

- Good agreement of central values
  (\( \lambda_B = 0.35 \text{ GeV} \))
Preference for small $\lambda_B$, i.e. strong spectator-scattering (as already evident in the global analysis of [MB, Neubert, '03]).
Charged $\pi^\mp \rho^\pm$ modes

$$R_3 = \frac{\Gamma(B_0 \rightarrow \pi^+ \rho^-)}{\Gamma(B_0 \rightarrow \pi^- \rho^+)}$$

$$\Delta C = \frac{1}{2} [C(\pi^- \rho^+) - C(\pi^+ \rho^-)]$$

Both quantities depend mainly on $f_\pi A_0^{B \rightarrow \rho}(0)/(f_\rho f_+^{B \rightarrow \pi}(0)) \times \alpha_1(\rho \pi)/\alpha_1(\pi \rho) = Re^{i\delta_T}$. Specifically,

$$\Delta C = \frac{1 - R^2}{1 + R^2} + \frac{4R^2}{(1 + R^2)^2} (a \cos \delta_a + b \cos \delta_b) \cos \gamma + \ldots$$

Seems to favour slightly smaller $B \rightarrow \rho$ to $B \rightarrow \pi$ form factor ratio than QCD sum rules ($\sim 1.25$). In the future: factorization test for $\rho \rho$ gives $B \rightarrow \rho$ form factor directly.
Look forward to NLO direct CP asymmetries to complete the picture.

2-loop corrections to phases can be large.