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Jan Wennekers will be missed across the world
He is survived by his wife and son
Lattice QCD simulations not (currently) at physical masses in a large volume. Interpolations are performed in $m_s$ 
Extrapolations are performed in $m_{ud}$ from larger masses $m_\pi \in [200 - 500] \, MeV$ down.

Some calculations follow unitary line $m_\pi^{\text{sea}} = m_\pi^{\text{valence}}$

Some calculations explore orthogonal direction and include unitary data as a subset: “partially quenched”

quenched $\neq$ unitary $\subset$ partially quenched

RBC-UKQCD approach has been to take large volume (self averaging) and maximise PQ-data in the chiral regime
Domain wall action

\[ D_{x,s;x',s'}^{\text{dwf}}(M_5, m_f) = \delta_{s,s'} D_{x,x'}^\parallel(M_5) + \delta_{x,x'} D_{s,s'}^\perp(m_f) \]

\[ D_{x,x'}^\parallel(M_5) = D_W(-M_5) \]

\[ D_{s,s'}^\perp(m_f) = \frac{1}{2} \left[ (1 - \gamma_5)\delta_{s+1,s'} + (1 + \gamma_5)\delta_{s-1,s'} - 2\delta_{s,s'} \right] \]

\[ - \frac{m_f}{2} \left[ (1 - \gamma_5)\delta_{s,Ls-1}\delta_{0,s'} + (1 + \gamma_5)\delta_{s,0}\delta_{Ls-1,s'} \right] . \]

(1)

- Different chiralities exponentially bound to each wall
- Adequately chirally symmetric for WME uses
- Rich phenomenology accessible \((B_K, K \rightarrow \pi\pi)\)
- Power of Lattice QCD is growing with time
- Importance of chiral fermions is growing with time
Published $24^3$ ensembles

Soon to be published $32^3$ ensembles

<table>
<thead>
<tr>
<th>$L^3 \times T \times L_s$</th>
<th>$(am_u,m_s)$</th>
<th>$\beta$</th>
<th>$a^{-1}$</th>
<th>L (fm)</th>
<th>$m_\pi$ (MeV)</th>
<th>$m_{\text{res}}$</th>
<th>$\tau$ MD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$24^3 \times 64 \times 16$</td>
<td>(0.005, 0.04)</td>
<td>2.13</td>
<td>1.73(2)</td>
<td>2.73</td>
<td>330</td>
<td>420</td>
<td>3.14 $\times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>(0.01, 0.04)</td>
<td></td>
<td></td>
<td></td>
<td>420</td>
<td>560</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.02, 0.04)</td>
<td></td>
<td></td>
<td></td>
<td>560</td>
<td>670</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.03, 0.04)</td>
<td></td>
<td></td>
<td></td>
<td>670</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$32^3 \times 64 \times 16$</td>
<td>(0.004,0.03)</td>
<td>2.25</td>
<td>$\sim 2.3$</td>
<td>$\sim 2.7$</td>
<td>$\sim 290$</td>
<td>$\sim 290$</td>
<td>$6.7 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>(0.006,0.03)</td>
<td></td>
<td></td>
<td></td>
<td>$\sim 350$</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>(0.008,0.03)</td>
<td></td>
<td></td>
<td></td>
<td>$\sim 400$</td>
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Additional three $16^3$ ensembles, one $48^3$ ensemble

Aux-det coarse $32^3$, 4.5 fm ensemble (aiming at $K \to \pi \pi$).
Summary of $24^3$ results

\[
\begin{align*}
    f &= 114.8(4.1)_{\text{stat}}(8.1)_{\text{syst}} \text{ MeV}, \\
    B_{\text{MS}}^{(2 \text{ GeV})} &= 2.52(0.11)_{\text{stat}}(0.23)_{\text{ren}}(0.12)_{\text{syst}} \text{ GeV}, \\
    \Sigma_{\text{MS}}^{(2 \text{ GeV})} &= \left(255(8)_{\text{stat}}(8)_{\text{ren}}(13)_{\text{syst}} \text{ MeV}\right)^3, \\
    \bar{t}_3 &= 3.13(0.33)_{\text{stat}}(0.24)_{\text{syst}}, \\
    \bar{t}_4 &= 4.43(0.14)_{\text{stat}}(0.77)_{\text{syst}}, \\
    \Lambda_3 &= 666(110)_{\text{stat}}(80)_{\text{syst}} \text{ MeV}, \\
    \Lambda_4 &= 1,274(92)_{\text{stat}}(490)_{\text{syst}} \text{ MeV}, \\
    m_{\text{MS}}^{\text{ud}}(2 \text{ GeV}) &= 3.72(0.16)_{\text{stat}}(0.33)_{\text{ren}}(0.18)_{\text{syst}} \text{ MeV}, \\
    m_{\text{MS}}^{\text{s}}(2 \text{ GeV}) &= 107.3(4.4)_{\text{stat}}(9.7)_{\text{ren}}(4.9)_{\text{syst}} \text{ MeV}, \\
    \tilde{m}_{\text{ud}} : \tilde{m}_{\text{s}} &= 1 : 28.8(0.4)_{\text{stat}}(1.6)_{\text{syst}}, \\
    f_\pi &= 124.1(3.6)_{\text{stat}}(6.9)_{\text{syst}} \text{ MeV}, \\
    f_K &= 149.6(3.6)_{\text{stat}}(6.3)_{\text{syst}} \text{ MeV}, \\
    f_K/f_\pi &= 1.205(0.018)_{\text{stat}}(0.062)_{\text{syst}}, \\
    B_{\text{MS}}^{(2 \text{ GeV})} &= 0.524(0.010)_{\text{stat}}(0.013)_{\text{ren}}(0.025)_{\text{syst}}. 
\end{align*}
\]
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RI-mom non-perturbative renormalisation
(cts-chart #5: 392 citations)

@Article{Martinelli:1994ty,
  title = "A General method for nonperturbative renormalization of lattice operators",
  journal = "Nucl. Phys.",
  volume = "B445",
  year = "1995",
  eprint = "hep-lat/9411010",
}
Non-perturbative renormalisation

DWF is ideal as off-shell improved!

Progress on $Z_m$ and $Z_{VV+AA}$ using the gauge-fixed off-shell Rome-Southampton RImom scheme.

Bilinear amputated vertex function

$$\Lambda(p, p')$$

in some scheme (Lattice or $\overline{\text{MS}}$) is related to the RI mom scheme defined at this momentum point by

$$Z_{\text{scheme}}^{\text{RI}} \Lambda_{\text{scheme}}(p, p') = \Lambda_{\text{tree}}(p, p')$$

Multi-loop calculations long available for $p = p'$ for bilinears $\mathcal{O}_{VV+AA}$ known only to 1-loop.

Conversion between Lattice and $\overline{\text{MS}}$ is then

$$Z_{\text{Lat}}^{\overline{\text{MS}}} = Z_{\text{Lat}}^{\text{RI}} / Z_{\text{MS}}^{\text{RI}} = \Lambda_{\overline{\text{MS}}}(p, p') / \Lambda_{\text{Lat}}(p, p')$$

For simultaneous discretisation and perturbative accuracy we require

$$\Lambda_{\text{QCD}}^2 \ll p^2 \ll \left( \frac{\pi}{a} \right)^2$$
Previously used point source propagators: now use plane wave source (Broemmel, Boyle)

\[ M(x, y)G(y) = \eta(x) = e^{ik_\mu x^\mu} \delta_{ij} \delta_{\alpha\beta} \]
\[ G'_{p1}(x) = G_{p1}(x)e^{-ip_1 \cdot x} = \sum_y M^{-1}(x, y)e^{ip_1 \cdot (y - x)} \]
\[ \Lambda_\Gamma = \langle \sum_z G'_{p1}(z) \rangle^{-1} \langle \sum_x \gamma_5 G'^\dagger_{p1}(x) \gamma_5 \Gamma G'_{p2}(x) \rangle \langle \sum_y G'_{p2}(y) \rangle^{-1} \]

Operator inserted at sink: \( L^4 \) volume average
Statistically precise – \( O_4 \) breaking now clear

\[
\sum_x \left( \gamma_5 (G'_{p1})^\dagger \gamma_5 \Gamma G'_{p2} \right) \left( \gamma_5 (G'_{p1})^\dagger \gamma_5 \Gamma G'_{p2} \right)
\]
Non-perturbative renormalisation

- RI-mom scheme traditionally uses $p = p'$
- Chiral symmetry $\Rightarrow \Lambda_A = \Lambda_V$
- Spontaneous $\chi^{SB}$ at low $p^2$
- 2\% systematic introduced; obscures good chiral properties of DWF

\[
\Lambda_A - \Lambda_V \text{ with } p = p'
\]

\[
\Lambda_A - \Lambda_V \text{ with non exceptional momenta } p^2 = (p')^2 = (p - p')^2
\]
Mass renormalisation

arXiv:0901.2599 (Sturm, Sachrajda, Aoki, Christ, Izubuchi)

From vertex function $\Lambda_S(p_1, p_2)$; $p_1^2 = p_2^2 = -\mu^2$ and $q^2 = -\omega\mu^2$ obtain

$$C_m = \frac{Z_{m}^{\text{MS}}}{Z_{m}^{\text{RI}-\omega}} = 1 - \frac{\alpha_s}{4\pi} C_{F} c_{m}^{(1)}(\omega)$$

$\omega = 1$ is the symmetric momentum point (smom)

$C_{m}^{\text{RI}} = 1 - \frac{\alpha_s}{4\pi} C_{F} 4 + \mathcal{O} (\alpha_s^2)$

$C_{m}^{\text{RIsmom}} = 1 - \frac{\alpha_s}{4\pi} C_{F} 0.4841391 + \mathcal{O} (\alpha_s^2)$

- Is this reduction preserved at higher orders?
- 5% perturbative error from RImom approach to $m_s$ could be greatly reduced
- How to estimate the perturbative error?
- Once anomolous running (SI-fit) included error may be larger

$$Z_{RI}^{MS}(\mu, p) = \frac{C_{RI}^{MS}(\mu)}{C_{RI}^{MS}(p^2)}$$
Non-exceptional $B_K$ chirality mixing consistent with zero

New: 1-loop conversion known for $Z_m, \mathcal{O}_{VV+AA}$
Sturm, Sachrajda arXiv:0901.2599 & preliminary

Higher order perturbative conversion to $\overline{MS}$ is needed

Jan Wennekers, RBC-UKQCD, preliminary

Five 1-loop mom schemes to estimate systematics

- Improved renormalisation may shift central value to around 0.54 from 0.52
- Expect good scaling from RBC-UKQCD data
- Two loop MOM scheme matching for non-exceptional kinematics required to reduce dominant error in the next few years.
$Z_{B_K}$ with non-exceptional momenta

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Unitary SU(2) NLO chiral expansion

\[
\begin{align*}
\chi_q &= 2Bm_q \\
M_P^2 &= \chi_q + \frac{\chi_q^2}{32\pi^2 F^2} \left[ \log \frac{\chi_q}{m_\pi^2} - \bar{l}_3 \right] \\
F_P^2 &= F - \frac{\chi_q}{16\pi^2 F^2} \left[ \log \frac{\chi_q}{m_\pi^2} - \bar{l}_4 \right] \\
F^{\pi\pi}(q^2) &= 1 + \frac{1}{F^2} \left[ -2l_6^r q^2 + 4\tilde{H}(\chi_q, q^2, \mu^2) \right] \\
\Pi_{V-A}^{(1)} &= -\frac{f^2}{q^2} - 8L_1^r (\mu) - \frac{\log \left( \frac{m_\pi^2}{\mu^2} + \frac{1}{3} - H(x) \right)}{24\pi^2} \\
\end{align*}
\]

Partially quenched more complex!
2+1 flavors: $SU(3)$ or $SU(2)$?

Debate: best mass extrapolation method for 2+1f?

- $SU(3)_{nf}$ – Treats the Kaon as a dynamical chiral pseudoscalar
  Least convergent terms expand in $(\frac{M_\eta}{4\pi f})^2$

  Treats the Kaon as a heavy spectator
  Least convergent terms expand in $(\frac{M_\pi}{M_K})^2$

- Analytic expansion around all physical quark masses BMW
  Discussed Lellouch, Kaon review Lattice 2008 arXiv:0902.4545
  Argument compelling near physical point

Debate:

- Inclusion of partial NNLO analytic terms?
- NNLO with non-lattice input for LEC’s?
- Finite volume correction of data?
- When to revert to polynomial extrapolation (Flavor expansion)?
SU3 vs SU2 with single lattice spacing

\[ f_{PS} \text{[MeV]} \]

\[ m_{ll} = 331 \text{ MeV} \]
\[ m_{ll} = 419 \text{ MeV} \]

\[ f_{ll}/f_0, \ m_h = 0.04 \]

Conclusions:
- Broad agreement that SU(2) is better for Kaons
- Disagreement about just how low \( F_0 \) is
- Convergence of \( SU(3) \chi^\text{PT} \) questioned 
  even for pionic lattice data
- Low prediction for \( f_\pi \) : discretisation or extrapolation?
Curious linearity of lattice data

Kelly, Mawhinney Lattice 2009.

- NLO SU(2) $\chi^{PT}$ extrapolates low in continuum limit
- Required NNLO terms are natural size
- Cannot fit *full* NNLO ; non-lattice input required
- Striking cancellation between NLO and NNLO for approx. linearity
- Analytic expansions for $f_\pi$ aren’t so bad after all!
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\[ \langle \pi(p')|V_\mu|K(p) \rangle = f_+(q^2)(p_\mu + p'_\mu) + f_-(q^2)(p_\mu - p'_\mu) \]

Several “double ratios” employed (hep-ph/0403217)

\[ \frac{\langle K(\vec{0})|V_0|\pi(\vec{0}) \rangle \langle K(\vec{0})|V_0|\pi(\vec{0}) \rangle}{\langle K(\vec{0})|V_0|K(\vec{0}) \rangle \langle \pi(\vec{0})|V_0|\pi(\vec{0}) \rangle} = \left( \frac{m_K + m_\pi}{4m_K m_\pi} \right)^2 |f_0(q_{\text{max}}^2)|^2 \]

Lattice methods undergone two related improvements

- Twisted boundary conditions enable simulation directly at \( q^2 = 0 \)
  Previously constrained to discrete Fourier momenta
  Model dependent interpolation to \( q^2 = 0 \) can be eliminated

- \( L^3 \) volume average can be taken improving statistical precision
Combined $q^2$ and chiral extrapolation w. hybrid pole dominance/Ademollo-Gatto constrained model.

$$f_0(q^2, m_{\pi}^2, m_K^2) = \frac{1 + f_2 + (m_K^2 - m_{\pi}^2)^2(A_0 + A_1(m_K^2 + m_{\pi}^2))}{1 - q^2/(M_0 + M_1(m_K^2 + m_{\pi}^2))^2}$$

$$f_+(0) = 0.9644(33)^{\text{stat}}(34)^{\text{extrapolation}}(14)^{\text{disc}}$$

- model dependence in $q^2$ interpolation & chiral extrapolation & strange mass adjust
- no continuum limit; budget 4% of $1 - f_+(0)$
Thanks to Zanotti, Juettner, De Lima

Recalculation for our lightest point ($m_l = 0.005$) using twisted BC approach ($Z_2$ wall noise sources; color-spin trace performed stochastically)

→ should better constrain chiral limit & address model dep. & stat errors

- Lines are prediction from global fit in PRL (black data)
  - *not* a fit to blue and red
- Global fit does correctly describe strange mass dependence
- Remove model dependence by direct $q^2 = 0$
- Improved statistical error with volume average (compare blue/red & black)

$32^3$ data nearly complete. Will address sub-dominant discretisation error.

Larger $f_2$ (range in $F_0$) will result in $\sim 0.5\%$ downward sys-error in our $f_+$ extrapolation.

Forthcoming paper nearly complete.
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\[ B_K = \frac{\langle K^0 | O_{VV+AA} | \bar{K}^0 \rangle}{\frac{8}{3} \langle K^0 | A_0 \rangle \langle A_0 | K^0 \rangle} \]

Chiral symmetry gives multiplicatively renormalised calculation

→ non-symmetric actions deal with unphysical taste/chirality mixings which inflate errors
Discrepancy between staggered $n_f = 0$ and $n_f = 2$ DWF has been noted as surprisingly large!

Already clarified by CP-PACS of DWF continuum limit

In quenched approximation JLQCD got a similar high staggered result
Has simply not agreed with simpler formulations in continuum limit.

One loop LPT staggered operator mixing is fragile

$N^{n_{\text{LO}}}$ lattice taste mixing easily underestimated & amplified through mixing matrix cancellation

Demonstration of universality of continuum limit is important!
Leading calculations of $B_K$ with 2+1 f.
Systematic error estimates are dominant, and a detailed breakdown of the sources error useful.
Two loop non-exceptional kinematic MOM calculation required

Matching uncertainty limits four best lattice calculations
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Matrix elements of BSM $\Delta_S = 2$ effective operators

$$\langle K_0|Q_i|K_0\rangle$$

\[
\begin{align*}
Q_1 & = s^a \gamma_\mu P_L d^a \bar{s}^b \gamma_\mu P_L d^b \\
Q_2 & = s^a \gamma_\mu P_L d^a \bar{s}^b \gamma_\mu P_R d^b \\
Q_3 & = s^a P_L d^a \bar{s}^b P_R d^b \\
Q_4 & = s^a P_L d^a \bar{s}^b P_L d^b \\
Q_5 & = s^a P_L d^a \bar{s}^b P_L d^b
\end{align*}
\]
Matrix elements of $Q_2, \ldots, Q_5$ allow $\epsilon_K$ to constrain certain BSM $\Delta_S = 2$ processes.

Mixing matrix becomes $5 \times 5$ in absence of chiral symmetry.

→ Overlap and DWF actions required for control.

Current best study remains quenched overlap hep-lat/0605016 Boston/Marseille/Wuppertal

SPQCD arXiv:0401033

$\beta=6.2$

- Standard RI-mom renormalisation has $\frac{1}{m p^2}$ infra-red pole.
- Residue in $p^2$ diverges as $m \to 0$
- Difficult to subtract precisely; affects all existing calculations
BSM bag parameters in 2+1f DWF

RBC-UKQCD (Wennekers) arXiv:0810.1841

- First calculation of SUSY bag parameters with dynamical chiral fermions
- Non-exceptional momentum eliminates pion pole
- Includes all lattice components of NPR
- Perturbative conversion to MS in progress (1-loop, Sachrajda & Sturm)
Bare matrix elements of $Q_2$ (left) and $Q_5$ (right) as function of mass

When 1-loop calculation is complete expect:

- matrix elements to 2% – comparable accuracy to RBC-UKQCD $B_K$
- Dramatic systematic improvement in renormalisation
Conclusions

Lattice QCD gives rigorous Kaon physics as all scales below cut-off.

RBC-UKQCD has exciting physics programme with chiral fermions
Broader phenomenology makes this the compelling long term approach

Chiral symmetry gives excellent off-shell non-perturbative renormalisation.
- Chiral ward identity protects $Z_A$
- Avoids unphysical mixing for V-A sector.
- Off-shell $O(a)$ improvement

Recent modifications improved RI-mom based approach (more to come!)
- Huge improvement in statistical precision
- Spontaneous chiral symmetry breaking suppressed

Badly need two-loop non-exceptional momentum $\overline{MS}$ calculations