Neutrino Mass, Dark Matter and Baryon Asymmetry via TeV-Scale Physics without Fine Tuning

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with

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12. Sep, 2008 Seminar at Southampton
Where is Toyama?

- It is located another side from Tokyo
- Japan Alps
- Much snow in Winter
- Lots of fishes
- U. of Toyama: Nearest University to Kamioka (Super-K)
What is discussed

• Although the success of the SM for long time, today we have definite reasons to consider new physics beyond the SM.
  – Neutrino oscillation
  – Evidence of Dark Matter
  – Baryon Asymmetry of the Universe

• In this talk, an extension of the SM is proposed, which can explain these phenomena without unnatural fine tuning.
Standard Model (SM)

- The Standard Model
  - Gauge Theory (SU(3) x SU(2) x U(1): Chiral for Matter)
  - Spontaneous Breakdown (Higgs Mechanism)
  - Flavor (Yukawa Interaction)

- Excellent description for the data for many years.

- Higgs remains unknown.
  - A SM-like Higgs boson may be found at the LHC if all goes well.
  - There is a possibility that Higgs takes a non-minimal form.

- We know phenomena of beyond the SM
  - Neutrino Data,
  - Dark Matter,
  - Baryon Asymmetry

- The Higgs sector may be a key for these problems.
1. Neutrino Oscillation

- **Information from Data**
  - Two Mass Scales \( \Delta m_{\text{sol}}^2 \sim 8 \times 10^{-5} \text{ eV}^2 \)
  - Mixing Angles \( \Delta m_{\text{atm}}^2 \sim 0.0021 \text{ eV}^2 \)
    \[ \theta_{\text{sol}} \sim 0.553 \]
    \[ \theta_{\text{atm}} \sim \pi/4 \]

- **In the SM, such phenomenon cannot be explained**
  - No Right-handed Neutrino in the SM \( (M_D=0) \)
  - No Source for Majorana Masses

- **New Physics**
  - Seesaw
  - Quantum Effect by Extended Higgs Sectors
Seesaw

• Super Heavy RH Neutrino \((M_{NR} \sim 10^{13-16}\text{GeV})\)
  – Hierarchy between \(M_{NR}\) and \(M_{D}\) generates that between \(M_{D}\) and tiny \(m_{\nu}\) \((M_{D} \sim 100\text{ GeV})\)
  \[m_{\nu} = \frac{m_{D}^2}{M_{NR}}\]
  – Simple, compatible with GUT

• Problem?
  – Has the problem really been solved ?
  Hierarchy for hierarchy !
  – Introduction of super high scale
  = far from experimental reach.

Yanagida
Gell-Mann et al
Quantum Effects

• Tiny $\nu$-Masses may come from loop effects
  – Zee (1980, 1985)
  – Zee-Babu
  – Ma (2006), ..... 

• Merit
  – Super large mass scales are not necessary
  – Tiny neutrino masses are radiatively generated
  No hierarchy problem

Physics at TeV: Testable at collider experiments
2. Cold Dark Matter

- Data from WMAP \( \Omega_{\text{DM}} h^2 \simeq 0.11 \)
- WIMP (Weakly Interacting Massive Particle)
  - Neutral and Stable
  - Cold DM \( M=10\text{GeV}-1\text{TeV} \)
- No candidate for CDM in the SM
- Stability may be guaranteed by a Discrete Symmetry
  - SUSY \( R\)-Parity
  - Little Higgs \( T\)-Parity
  - Krauss et al. \( Z_2\)-Parity

- Thermal Relic abundance of DM:
  \[ \Omega h^2 \propto \left(\text{Rate of Pair-Annihilation}\right)^{-1} \]
3. Baryon Asymmetry of Universe

• Baryogenesis
  \[ n_B/s = (9.2 \pm 1.1) \times 10^{-11} (WMAP) \]

• Sakharov’s 3 conditions:
  – Baryon number violation
  – C, and CP violation
  – Departure from thermal equilibrium

• Scenarios for baryogenesis
  – B-L generation above the EW phase transition (Leptogenesis etc).
  – B+L gen. at the EW phase transition (EWBG)
    EWBG can be tested at collider experiments
    • Strong 1\textsuperscript{st} Order EW Phase Transition
    • CP violation

• In the SM, EWBG scenario requires \( m_h < 60 \text{ GeV} \), so that it is excluded
Motivation of our model

• Is it possible to extend the SM to include
  – Neutrino Masses
  – DM
  – Baryon Asymmetry of the Universe

in the framework of a renormalizable field theory of at most TeV scale?

• No more large mass scales

• No more unnatural fine tuning among coupling constants,...
The answer is Yes

We can construct a model to explain

- Mass of neutrinos  [Z2 parity, extended Higgs, RHν]
- DM  [Z2 Parity, extended Higgs]
- Baryogenesis  [extended Higgs (EWBG)]

as a renormalizable field theory of at most 1 TeV.

Assumptions:

1. No unnatural fine tuning among couplings
2. No large hierarchy in mass scales
3. Charged lepton (e, μ, τ) mass hierarchy is an input.
4. TeV scale RH neutrinos as source of lepton number generation
Hierarchy problem?

• Our Purpose
  – To construct a new model below Tera-eV scale that can solve the three new phenomena without assuming large hierarchy among mass scales.

• Naturalness Problem is not a big issue here.
  – Quadratic Divergences in SM Higgs
    Hierarchy problem, if the SM holds until very high energies.
  – This is not a problem if the cutoff scale is lower than multi TeV -10 TeV.
  – Still we can consider New Physics at TeV as a low energy effective theory of UV complete theories.
The Model

- **Exact $Z_2$ Parity**
  - No neutrino Yukawa
  - Tiny $\nu$-mass: 3-loop induced
  - Stabilization of Dark Matter
- **Extended Higgs**: $2HDM + S^+ + \eta^0$
  - Tiny $\nu$-mass: Close $Z_2$-odd loop $(S^+, \eta^0)$
  - 3-loop induced $(H^+)$
  - EW Baryogenesis [$1^{st}$ Order PT, Source of CPV]($2HDM$)
Symmetries of the model

- Gauge Symmetries \( SU(3) \times SU(2) \times U(1) \)

- Discrete symmetry \( Z_2 \) (Exact)

- In general 2HDM \( \Phi_1, \Phi_2 \rightarrow \text{FCNC}! \)

   Another discrete symmetry is necessary:
   \[
   \tilde{Z}_2 : \Phi_1 \rightarrow + \Phi_1, \quad \Phi_2 \rightarrow - \Phi_2,
   \]
   \[
   e_R \rightarrow + e_R, \quad L_L \rightarrow + L_L
   \]

   Only \( \Phi_1 \) couples to charged leptons.
   This can be softly broken.

<table>
<thead>
<tr>
<th></th>
<th>( Q^i )</th>
<th>( u^i_R )</th>
<th>( d^i_R )</th>
<th>( L^i )</th>
<th>( e^i_R )</th>
<th>( \Phi_1 )</th>
<th>( \Phi_2 )</th>
<th>( S^\pm )</th>
<th>( \eta )</th>
<th>( N^\alpha_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_2 ) (exact)</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \tilde{Z}_2 ) (softly broken)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>
Lagrangian

\[ 2\text{HDM} + Z_2^{\text{odd}}(S^+, \eta^0, N_R^\alpha) \]

\[ \mathcal{L}_Y = - \sum_{\alpha=1}^{2} \sum_{i,j=1}^{3} \bar{h}_i^\alpha (e_R^i)^c N_{R}^\alpha S^- + \sum_{\alpha=1}^{2} m_N^\alpha N_{R}^c N_{R}^\alpha + \text{h.c.} \]
Physical States

- Exact $Z_2$ parity: even and odd states do not mix.
- Masses of 2HDM fields can be diagonalized by the mixing angles $\alpha$ and $\beta$ as usual.

\[
\Phi_i = \begin{bmatrix}
\frac{1}{\sqrt{2}} (v_i + h_i + iz_i) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
w_1^+ \\
w_2^+
\end{bmatrix} = \begin{bmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{bmatrix}
\begin{bmatrix}
w_1 \\
H^+
\end{bmatrix}
\]

\[
\begin{bmatrix}
h_1 \\
h_2
\end{bmatrix} = \begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
H \\
h
\end{bmatrix}
\]

- $Z_2$-even physical states
  - $h$ (SM like Higgs) for $\sin(\beta-\alpha)=1$  \( g_{hWW}=1 \)
  - $H, A, H^-$ (Extra scalars)

- $Z_2$-odd states
  - $\eta, S^+, N_R^\alpha$
  - $g_{HWW}=0$
Neutrino Mass ($\nu\nu\phi\phi$)

Tree level neutrino Yukawa is forbidden by $Z_2$

Neutrino mass matrix is generated at the 3-loop level.

\[ M_{ij} = \sum_{\alpha=1}^{2} C_{ij}^{\alpha} F(m_H, m_S, m_{N_R^\alpha}, m_\eta) \]

\[ C_{ij}^{\alpha} = 4\kappa^2 \tan^2 \beta (y_{\ell i}^{SM} h_i^{\alpha})(y_{\ell j}^{SM} h_j^{\alpha}) \]

$Z_2$-even physical states
- $h$ (SM like Higgs)
- $H$, $A$, $H^-$ (Extra scalars)

$Z_2$-odd states
- $\eta$, $S^+$, $N_R^\alpha$
Neutrino Masses

\[ M_{ij} = \sum_{\alpha=1}^{2} C_{ij}^\alpha F(m_H, m_S, m_{N_R^\alpha}, m_\eta) \]

Mixing Structure is determined by

\[ C_{ij}^\alpha = 4\kappa^2 \tan^2 \beta (y_{e_i}^{SM} h_i^\alpha)(y_{\ell_j}^{SM} h_j^\alpha) \]

Universal scale is determined by the 3-loop factor \( F \)

\[ F(m_{H^\pm}, m_{S^\pm}, m_{N_R}, m_\eta) = \left( \frac{1}{16\pi^2} \right)^3 \frac{(-m_{N_R} v^2)}{m_{N_R}^2 - m_\eta^2} \]
\[ \times \int_0^\infty dx \left[ x \left\{ B_1(-x, m_{H^\pm}, m_{S^\pm}) - B_1(-x, 0, m_{S^\pm}) \right\} \right]^2 \]
\[ \times \left( \frac{m_{N_R}^2}{x + m_{N_R}^2} - \frac{m_\eta^2}{x + m_\eta^2} \right) \quad , \quad (m_{S^\pm} \gg m_{e_i}^2) \]
Numerical Evaluation

1. LFV data                           \( N_R \) must be \( O(1) \) TeV
2. \( \nu \) data                      Then, \( m_{H^+} < O(100) \) GeV
3. LEP direct search on \( H^+ \)       \( m_{H^+} > 90 \) GeV
4. LEP precision measurement          \( [\rho \text{ parameter}] \)
   \[ \sin(\beta - \alpha) = 1, \ m_{H^+} = m_H \]

From natural assumption \( \kappa \tan \beta < O(10), \ h^\alpha_{e} = O(1), \)
possible parameters are uniquely determined as
\[ \sin(\beta - \alpha) = 1 \quad (h \text{ is the SM-like Higgs}), \]
\[ m_{H^+} = m_H = 100 \text{GeV}, \ m_S = O(100) \text{ GeV} \]
\[ m_N = 3 \text{ TeV} \]
Mass and mixing

\[ M_{ij} = U_{is} (M_{\nu}^{\text{diag}})_{st} (U^T)_{tj} \]

\[ m_{\nu}^{\text{diag}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\Delta m^2_{\text{solar}}} & 0 \\ 0 & 0 & \sqrt{\Delta m^2_{\text{atm}}} \end{bmatrix} \]

\[ U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13} e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{-i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{bmatrix} \]

\[ C_{ij}^\alpha = 4 \kappa^2 \tan^2 \beta (y_{\ell_i}^{\text{SM}} h_i^\alpha) (y_{\ell_j}^{\text{SM}} h_j^\alpha) \]

<table>
<thead>
<tr>
<th>Set</th>
<th>( h_e^1 )</th>
<th>( h_e^2 )</th>
<th>( h_\mu^1 )</th>
<th>( h_\mu^2 )</th>
<th>( h_\tau^1 )</th>
<th>( h_\tau^2 )</th>
<th>( B(\mu \rightarrow e\gamma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.2</td>
<td>1.2</td>
<td>-0.011</td>
<td>0.025</td>
<td>-0.0015</td>
<td>0.00070</td>
<td>5.3 \times 10^{-12}</td>
</tr>
<tr>
<td>B</td>
<td>1.2</td>
<td>1.35</td>
<td>0.0037</td>
<td>0.022</td>
<td>-0.00075</td>
<td>0.0012</td>
<td>4.5 \times 10^{-12}</td>
</tr>
</tbody>
</table>

\( m_{H^+} = m_{H} = m_{S} = 100 \) GeV, \( m_{\eta} = 50 \) GeV, \( m_{\text{NR}}^1 = m_{\text{NR}}^2 = 3.5 \) TeV

Set A (B): \( \kappa \tan \beta = 36 \) (42) and \( U_{e3} = 0 \) (0.18).
Type-X Yukawa coupling

• In our model, a light charged Higgs boson is predicted.
• In the type II 2HDM (MSSM), such a light H+ is excluded because of $b \rightarrow s \gamma$ result
• We employ alternative Yukawa coupling (type-X)
• Type-X 2HDM

$$\mathcal{L}_Y = -y_{e_i} \bar{L}^i \Phi_1 e^i_R - y_{u_i} \bar{Q}^i \Phi_2 u^i_R - y_{d_i} \bar{Q}^i \Phi_2 d^i_R + \text{h.c.}$$

– $\Phi_1$ couples to Leptons
– $\Phi_2$ couples to Quarks

• Discriminative Higgs phenomenology

  B-physics limit on extra Higgs be relaxed!
  Leptonic decays be dominant!
The NLO calculation and the data

M. Aoki, S.K., K. Tsumura, K. Yagyu, in preparation
Thermal Relic Abundance of $\eta^0$

Annihilation Cross Sections determine the abundance

$$\Omega_{\eta} h^2 = 1.1 \times 10^9 \left( \frac{m_\eta / T_d}{\sqrt{g^*_T} M_P \langle \sigma v \rangle} \right)_{T_d} \text{ GeV}^{-1}$$

Both $b\bar{b}$ and $\tau\tau$ included

$m_\eta$ would be around 49-63 GeV

$\eta$: DM candidate
Baryogenesis mechanism

- Asymmetry of the charge flow of the particle (due to \( CP \) violation)

- Accumulation of the charge in the symmetric phase

- \( B \) generation via sphaleron process

- Decoupling of sphaleron process in the broken phase

- Strongly 1st order phase transition

  \[ \Gamma_{\text{sph}}^{(b)} / T_c^3 < H(T_c) \implies \frac{\varphi_c}{T_c} \gtrsim 1 \]
Finite temperature Higgs potential

For $m^2_{\Phi}(v) \gg M^2, m^2_{h}(v)$ \quad $m^2_{\Phi}(\varphi) \simeq m^2_{\Phi}(v)\frac{\varphi^2}{v^2}$, \quad ($\Phi = H, A, H^\pm$)

\[ V_{\text{eff}} \simeq D(T^2 - T_0^2)\varphi^2 - ET\varphi^3 + \frac{\lambda_T}{4}\varphi^4 \]

where \quad $E = \frac{1}{12\pi v^3}(6m^3_W + 3m^3_Z + \underbrace{m^3_H + m^3_A + 2m^3_{H^\pm}}_{\text{additional contributions}})$

At $T_c$, degenerate minima: \quad $\varphi_c = 0$, \quad $\frac{2ET_c}{\lambda_T}$

- The magnitude of $E$ is relevant for the strongly 1st order phase transition

- We examine the strength of the phase transition without the high temperature expansion.
**Strong 1\textsuperscript{st} Order Phase Transition**

- For sphaleron decoupling in the broken phase:
  \[
  \frac{\varphi_c}{T_c} \left( = \frac{2E}{\lambda T_c} \right) \gtrsim 1
  \]
  \[
  \lambda T \sim \frac{2m_h^2}{v^2}
  \]

- SM satisfy this only for a light Higgs (Excluded).
  \[m_h < 50-60\text{ GeV}\]

- In our model, it is satisfied for \( m_h = 120\text{ GeV} \), because of larger \( E \).

We require non-decoupling effect of the Higgs sector.

Mass difference between \( A \) and \( H^+ \)
Phenomenological Constraints

- **Model Parameters**
  - \( h: \text{SM like } \sin(\alpha-\beta)=-1 \)
  - New Couplings \( \kappa, h_i^\alpha < O(1) \)

- **LEP Bounds (Direct Search, Indirect Search)**
  - Custodial Symmetry \( \sin(\beta-\alpha)=1, m_H=m_{H^+} \)
  - \( m_h=120 \text{ GeV} \)
  - \( m_{H^+}(=m_H) > 100 \text{ GeV}, m_S > 100 \text{ GeV} \)

- **Neutrino Data**
  - \( m_{H^+} < 120 \text{ GeV}, m_s < 200 \text{ GeV}, \tan\beta > 3 \)

- **LFV Limit**
  - \( m_{NR} \sim 3 \text{ TeV} \)

- **B Physics Bounds**
  - No bound (because of Type X)

- **Thermal Relic Abundance**
  - \( m_\eta = 47-62 \text{ GeV} \)

- **Strong 1\textsuperscript{st} Order PT**
  - \( m_A > 300 \text{ GeV} \)

Most of Parameters are constrained by the current Data
→ Predictive Power at future experiments.
Mass Spectrum

The current data gives tight constraints on the masses

- RHν: \( O(1) \) TeV
- Strong coupled A: \( m_A > 350 \) GeV
- Multi Higgs: 100-200 GeV
- DM (\( \eta \)): around 60 GeV

All the masses are \( O(100) \) GeV – \( O(1) \) TeV
Prediction

• Mass spectrum
• Higgs invisible decays into DM
• Detectable at DM searches
• A large hhh coupling (non-decoupling effect)
• Unique features in H, H+, A decays (Type-X)
• Light Higgs scenario (h, H, H+)
• Charged Higgs phenomenology (H+, S+)
DM physics

Physics of $\eta$

- $h$ is the SM-like Higgs boson but decays into $\eta \eta$

$$B(h \rightarrow \eta \eta) = 50\% \ (37\%)$$

for $m_\eta = 48 \ (57) \ \text{GeV}$

Indirectly testable via the invisible Higgs decay at the LHC

- $\eta$ from the halo can basically be detected at the direct DM search (CDMS, XMASS)

\[ \begin{array}{c}
\eta \\
\hline
Xe, Ge
\end{array} \]

Observing the release energy
Non-decoupling property

- EWBG requires a large mass splitting between $m_A$ and $m_{H^+}$

$$m_A^2 - m_{H^+}^2 = (\lambda_4 - \lambda_5)v^2$$

Strong 1\textsuperscript{st} Order EWPT

- This effect enhances the $hhh$ coupling by 20-30\%.

- It is detectable at the ILC
Extra Higgs (H, H+, A) phenomenology
– Branching Fraction completely different

Model II (MSSM) vs Type X

\sin(\alpha - \beta) = -1, \ m_h = 120\text{GeV}, \ m_H = m_A = 160\text{GeV}
Extra Higgs (H,H+,A) phenomenology
– Light Higgs scenario: Production at the LHC

$$pp \rightarrow W^\pm \rightarrow HH^+ (AH^+)$$

$$HH^+ \rightarrow (\tau\tau)(\tau\nu)$$

$$AH^+ \rightarrow (W^\pm H^\mp)(\tau\nu) \rightarrow jj(\tau\nu)(\tau\nu)$$

(MSSM) $$pp \rightarrow AH^+ \rightarrow (b\bar{b})\tau^+\nu \rightarrow (b\bar{b})(\pi^+\bar{\nu}\nu)$$

Pions from $H^+\rightarrow\tau\nu$ are harder than those from $W^+\rightarrow\tau\nu$

High energy pions

low energy pions

Cao, SK, Yuan
Bullock, Hagiwara, Martin
• $Z_2$-odd charged scalar $S^+$
  - Produced in pair
    $$e^+ e^- \rightarrow S^+ S^-$$
  - Signal should be hard pions with large missing energy
    $$S^\pm \rightarrow H^\pm \eta \rightarrow \tau^\pm \nu \eta \rightarrow \pi^\pm \nu \eta$$
  - Indirect quantum effect can be large

\[ \text{Graphical diagrams showing production and decay processes.} \]

SK, Lin, Kasai, Okada, Yuan
Summary

• Phenomena, which the SM cannot solve
  – Neutrino oscillation
  – Dark Matter
  – Baryon Asymmetry of Universe
• We construct a model to solve these problems by TeV-scale physics \((\Phi_1, \Phi_2, \eta, S^+, N_R)\)
  Extended Higgs sector
  RH neutrinos (whose masses are at TeV)
  \(Z_2\) Parity
• The model gives many discriminative predictions in Higgs physics, LFV and DM physics
  – Testable at Collider Experiments (LHC, ILC)
  – DM also testable by direct and indirect search
  – LFV
• Further phenomenological study is underway.
Masses are determined by vev and M (or $\mu_{s,\eta}$)

\[
\begin{align*}
m_{h}^2 &= O(\lambda) \nu^2 \quad (\text{SM like: } \sin(\beta-\alpha)=1) \\
m_{H}^2 &= M^2 + O(\lambda) \nu^2 \\
m_{A}^2 &= M^2 + O(\lambda) \nu^2 \\
m_{H^+}^2 &= M^2 + O(\lambda) \nu^2 \\
m_{S^+}^2 &= \mu_{S^+}^2 + O(\rho) \nu^2 \\
m_{\eta}^2 &= \mu_{\eta}^2 + O(\sigma) \nu^2
\end{align*}
\]

\[
M = \frac{|\mu_{12}|}{\sqrt{\sin \beta \cos \beta}}
\]

Soft breaking scale for $\tilde{Z}_2$
CP violating phases

- In Higgs potential $m_3^2$ and $\lambda_5$ are complex, that cause CP violation.
- Although the CP phase is crucial for generating baryon number, it does not affect much in the discussions on $m_\nu$, DM and 1st Order EWPT.
- We neglect it for simplicity
- Later comment on the case including it.
Neutrino Masses from Higgs Sector

Quantum Effect by EW (TeV) physics

- **Zee Model**  \( D+D+S^+ \)
  
  No RH-\( \nu \)
  
  \( S^+ \) carries \( L=2 \)
  
  1-loop induced

- **Krauss et al. Model**  \( D+S^++S^++NR \)
  
  \( Z_2 \) symmetry
  
  \( Z_2 \) odd
  
  3-loop induced
  
  (More Natural for neutrino masses)
  
  \( N_R \) can be a DM candidate

Two generation of \( N_R \) explains the mixing

Cheung, Seto
\( \Delta m_{\text{sol}}^2 \sim 8 \times 10^{-5} \text{ eV}^2 \)

\( \Delta m_{\text{atm}}^2 \sim 0.0021 \text{ eV}^2 \)

\( \theta_{\text{sol}} \sim 0.553 \)

\( \theta_{\text{atm}} \sim \pi/4 \)